博士論文

Investigation of Phase Transitions in Quark-Hadron Many-Body Systems using an Extended NJL Model with Scalar-Vector Interactions

（スカラー・ベクトル相互作用を含む拡張されたNJL模型を用いたクォーク・ハドロン多体系の相転移の研究）

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Investigation of Phase-Transitions in Quark-Hadron Many-Body Systems using an Extended NJL Model with Scalar-Vector Interaction

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Abstract

Understanding various aspects of quark-gluon and hadronic many-particle systems interacting via strong interactions described by quantum chromodynamics (QCD) under multi-extreme conditions (e.g., high temperature, high density, strong magnetic field, etc.) is one of the most fascinating subjects in modern theoretical physics. Above all, unraveling the phase structure of quark-hadron many-body systems on the temperature-chemical potential plane is the most recent topics of interest, which leads to the understanding and development of the physics in the early universe and the neutron star. The RHIC/LHC-energy experiments and the lattice-QCD simulations known as a Monte Carlo analysis are accessible to the high-temperature and low-density regime. As for the high-density regime, however, some experiments to create the dense QCD matter are now in preparation or under construction. Also, the finite-density lattice-QCD simulation is infeasible because the fermion determinant is complex and hence its real part can be negative, which is called the sign problem. Thus, the heavy ion collisions and the lattice calculations alone are insufficient to understand the finite density QCD, so that an effective model approach based on QCD would be a useful tool to deal with finite density systems. This leads to a gain of qualitative insights of intermediate-density regime. In this thesis, we therefore concentrate on the finite density systems.

The focus of this thesis is to study of the quark-hadron phase transition at finite temperature and baryon chemical potential with a Nambu-Jona-Lasinio (NJL)-type effective model. Here, we adopt an extended NJL model with a reasonable saturation property as a model for nuclear matter and an extended NJL model with an additional interaction as a model for quark matter, respectively. We apply the above models including the scalar-vector eight-point interaction to both symmetric nuclear matter and free quark matter. As a result, the chiral and quark-hadron phase boundaries are obtained at finite temperature and baryon chemical potential by comparing the pressures of nuclear and quark matters, i.e., by using three equations of motion for the quark phase with dynamical mass, the massless quark phase, and the nuclear phase. Also, a quarkyonic-like phase, in which the chiral symmetry is restored but the elementary excitation modes are nucleonic, appears just before deconfinement in this model. Furthermore, the scalar-vector coupling dependence of phase diagram is summarized in this thesis.
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Chapter 1

Introduction

1.1 Quantum chromodynamics

Strongly correlated quantum many-body systems such as quark-hadron matter are described
by quantum chromodynamics (QCD) which is a fundamental theory of strong interactions. The
QCD is a non-Abelian gauge theory with color \(SU(3)\) symmetry in which the fundamental degrees
of freedom are quarks belonging to the fundamental representation of \(SU(3)\) and gluons belonging
to the adjoint representation of \(SU(3)\). In this section, we briefly introduce the basics of QCD
and its important features such as non-perturbative aspects of the QCD vacuum structure. For
details, e.g., see Refs. [1] [2]. In addition, an overview of the QCD phase diagram exploration and
its experimental and theoretical approaches are also outlined [3] [4].

1.1.1 Basic concepts and phenomenological aspects of QCD

The gauge invariant QCD Lagrangian density\(^1\) is given by

\[
\mathcal{L}_{\text{QCD}} = \bar{\psi}_q \left[ i \gamma^\mu (\partial_\mu - ig A^a_\mu \gamma^a) - m^f_q \right] \psi^f_q
\]

\[
- \frac{1}{4} \left( \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu \right) \left( \partial^\mu A^a_\nu - \partial^\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \right),
\]

where the gauge field \(A^a_\mu\) denotes the gluon field, and the Greek and Roman letters represent
the Lorentz and color indices in which the repeated ones are summed up, respectively. Here, \(\psi^f_q\)
and \(m^f_q\) are the quark field and the current quark mass (bare mass) in the flavor \(f\), respectively.
Also, \(g\) is a dimensionless coupling constant, the so-called QCD coupling constant, and \(\gamma^\mu\) and
\(f_{abc}\) are the Dirac gamma matrices and the structure constant of \(SU(3)\) Lie algebra, respectively.
The \(SU(3)\) generators \(\{t^a\}\), satisfying the commutation relation \([t^a, t^b] = if_{abc} t^c\), is normalized

\(^1\)This is the so-called classical QCD Lagrangian density. In the quantized QCD Lagrangian density, the gauge fixing
and Faddeev-Popov ghost terms are introduced. Naively, the QCD Lagrangian density has the CP violating term or
the theta term which is proportional to \(\tilde{\theta}_{\mu
\nu
\rho
\sigma}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu)(\partial^\rho A^a_\sigma - \partial^\sigma A^a_\rho + g f^{abc} A^b_\rho A^c_\sigma)\). However,
the parameter \(\tilde{\theta}\) is estimated to be \(|\tilde{\theta}| < 10^{-9}\) from the neutron electric dipole moment (nEDM) measurements.
Thus, we ignored the theta term since \(\tilde{\theta}\) is sufficiently small. Incidentally, this smallness of \(\tilde{\theta}\) is not well-known. This
puzzle is called the strong CP problem [5].
as Tr$(t^a t^b) = \delta^{ab}/2$ in the fundamental representation. Under the $SU(3)$ gauge transformation, i.e., $\psi^a_j \to U(x) \psi^a_j$ and $A^a_\mu \to U(x)(A^a_\mu(x) + i \partial_\mu)U^\dagger(x)$ where $U(x) = \exp(i\theta^a(x)t^a)$ with the transformation parameter $\theta^a(x)$, the Lagrangian density (1.1) is invariant in the color space.

The QCD is renormalizable, so that the bare coupling constant $g$, as well as the bare mass $m_f$, depends on the energy scale $Q$. Thus, by considering the renormalization group equation based on the perturbative calculation, the coupling $g$ becomes an energy scale dependent coupling constant known as the running coupling constant. This coupling $g$ is effectively equivalent to be $g^2 \to g^2 [1 - g^2 b_0 \ln(Q^2/\kappa^2)]$ where $Q^2$ and $\kappa^2$ denote the square of the Euclidean four-momentum and that of the renormalization point, respectively. Unlike quantum electrodynamics (QED), in QCD, there exist not only interactions between quarks and gluons but also gluon self-interactions with three- and four-gluon vertices. As a result, up to one-loop order, the running coupling constant $\alpha_s(Q^2)$ is written as

$$\alpha_s(Q^2) = \frac{g_{\text{eff}}^2(Q^2)}{4\pi} = \frac{1}{2\pi b_0 \ln(Q^2/\Lambda_{\text{QCD}}^2)},$$

(1.2)

where $g_{\text{eff}}^2(Q^2) = g^2/[1 + g^2 b_0 \ln(Q^2/\kappa^2)]$ and $b_0 = \frac{11N_c - 2N_f}{24\pi^2}$ with the number of colors $N_c$, being three, and that of quark flavors $N_f$. The QCD scale parameter $\Lambda_{\text{QCD}} = \kappa \exp(-1/b_0g(\kappa^2))$ is experimentally determined and its current estimate is that $\Lambda_{\text{QCD}} \approx 200$ MeV ($\approx 1$ fm$^{-1}$) for five flavors in the modified minimal subtraction (MS) scheme. In the year 2009, for the coupling strength of strong interactions, a measurement value is evaluated to be around 0.12 at the Z-boson mass [6]. For the latest version, see e.g., Ref. [7] in which the overall result is unchanged from the value obtained in 2009. In the high-energy regime (short-distance regime), the running coupling constant decreases with $\ln(Q^2/\Lambda_{\text{QCD}}^2)^{-1}$, i.e., the interaction between quarks and gluons decreases with increasing energy scale, in which $N_f \leq 16$. This property is called asymptotic freedom [8,9]. On the other hand, in the low-energy regime (long-distance regime), the coupling constant grows, i.e., the interaction between quarks becomes strong. However, the coupling $\alpha_s(Q^2)$ diverges near the QCD scale, $Q^2 \sim \Lambda_{\text{QCD}}^2$, so that the perturbation theory fails. This feature may be related to the phenomenon called color confinement in which color-charged particles such as quarks and gluons cannot be isolated singularly.

As for the confinement phenomenon, it is possible to consider a linear confining potential as an effective potential with light quarks in the long-distance regime. This approach has been estimated since a long time by using the linearity of Regge trajectories [10]: $J = \alpha + \alpha' M^2$ where $M, J, \alpha$, and $\alpha'$ represent the mass of hadron, its spin, the intercept of Regge trajectory, being around zero, and the slope of Regge trajectory, being about 1 GeV$^2$, respectively. The Regge spectra provide the most striking evidences for the string model in which the quark and antiquark are connected by a string for a meson, and suggests that the slope $\alpha'$ is concerned with the string tension: $\sigma' = (2\pi\sigma)^{-1}$ where $\sigma$ is the string tension [11]. As a result, in the phenomenological potential model of QCD, the effective potential at low energy becomes to be linear and the force given by this potential is a distance-independent attractive force. Thus, in the long-distance regime, it takes an infinite amount of energy to pull out an isolated quark from the hadrons, in which the quark-antiquark pairs are produced. In contrast, in the short-distance regime, the potential is Coulomb-like in which the
coupling constant becomes to be weak-coupling-like one from the asymptotic freedom. Therefore, the effective potential, written as $V_{\text{eff}}(r)$, consists of a Coulomb-like part and a confining part: $V_{\text{eff}}(r) = -4\alpha_s(r)/3r + \sigma r$ with $\alpha_s(r) = g_{\text{eff}}^2/4\pi$ where $r$ is a distance between the quarks. Here, the coefficient of $-1/r$ in the first term of $V_{\text{eff}}(r)$ represents a strength of the Coulomb attraction, which reflects the asymptotic freedom, and the coefficient $\sigma$ in the last term of $V_{\text{eff}}(r)$ is the string tension which denotes a strength of the attractive color force.

In the low-energy regime of QCD, the perturbative expansion is not available due to the strong-coupling effects, so that the non-perturbative phenomena beyond the perturbation theory emerge. As a result, the QCD vacuum structure is converted to the non-trivial one. For instance, the quark confinement becomes conspicuous as one of the non-perturbative aspects of the QCD vacuum, in which the quarks and gluons interact non-perturbatively. Also, the QCD vacuum is characterized by non-zero quark and gluon condensates. In case that the quark mass is small, for the quark condensate, one finds $\langle \bar{q} q \rangle_0 \approx -(250 \text{ MeV})^3$ by using the Gell-Mann-Oakes-Renner (GOR) relation, i.e., $f_{\pi}^2 M_{\pi}^2 = -2 M_q \langle \bar{q} q \rangle_0$ where $f_{\pi}$ is the pion decay constant whose observed value is about 93 MeV, $M_{\pi}$ is the charged pion mass whose observed value is about 140 MeV, and $M_q = (m_u + m_d)/2$ is the averaged mass of $u$ and $d$ quarks whose value is around 5 MeV. Here, we assumed the value of $M_q$ where the values obtained by the running masses of light quarks and the vacuum expectation values $\langle \bar{q} q \rangle_0 = \langle \bar{q}_u q \rangle_{\text{vac}} = \langle \bar{q}_d q \rangle_{\text{vac}}$ are used. As for the gluon condensate, from the analyses of the charmonium mass spectrum by using the QCD sum rules, one finds $\langle \bar{q} q F_{\mu\nu} F^{\mu\nu} \rangle_0 \approx (300 \text{ MeV})^4$ where $F_{\mu\nu}$ is the field strength tensor defined as $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + g f_{abc} A_{\mu}^b A_{\nu}^c$.

Another non-perturbative aspects of the QCD vacuum is the spontaneous chiral symmetry breaking. The chiral symmetry, albeit approximately, is the important symmetry of QCD\footnote{Although the current quark mass is actually non-zero, the chiral symmetry may be considered as an approximate symmetry in QCD Lagrangian since the light quarks are very small compared with hadronic scale.\footnote{In this thesis, we consider the two-flavor case, i.e., only $u$ and $d$ quarks are treated. Thus, the influence of heavy quark is ignored here.}}, in which the Lagrangian is invariant under the flavor $\mathcal{U}_L(N_f) \otimes \mathcal{U}_R(N_f)$-global transformation: $\psi_q \rightarrow \exp(i\theta_q^a \gamma^a) \psi_q$ where $\psi_q^{R/L} = (\frac{1 + \gamma_5}{2} \psi_q)$ are the right-handed (R) and left-handed (L) quark fields which are the eigenstates of the chirality operator $\gamma_5$. Here, $\theta_q^a_{L/R}$ are the transformation parameters ($\theta_q^a_{L}$ and $\theta_q^a_{R}$ are independent of each other), and $\{T^a\}$ is $U(N_f)$ generators with $a = 1, \ldots, N_f^2 - 1$. In the limit of vanishing quark masses, the classical QCD Lagrangian has the $SU(N_f)_L \otimes SU(N_f)_R$-chiral symmetry.

For simplicity, we consider the two-flavor case\footnote{In this thesis, we consider the two-flavor case, i.e., only $u$ and $d$ quarks are treated. Thus, the influence of heavy quark is ignored here.}. The massless QCD Lagrangian is invariant under the $SU(2)_L \otimes SU(2)_R$-chiral transformation and equivalent to be invariant under the global vector transformation, i.e., $SU_V(2) : \psi_q \rightarrow \exp (i\frac{\theta_V^a}{2} \gamma^a) \psi_q$ with $\theta_V = \theta_L = \theta_R$, and the axial-vector one, i.e., $SU_A(2) : \psi_q \rightarrow \exp (i\gamma_5 \frac{\theta_A^a}{2} \gamma^a) \psi_q$ with $\theta_A = -\theta_L = \theta_R$. The transformations $SU_V(2)$ and $SU_A(2)$ represent the two-component spinor transformations in the Yang-Mills theory, in other words, the isospin transformation which corresponds to the expression of the right- and left-handed systems. Also, the corresponding conserved currents are the vector current, i.e., $J_V^a = \bar{q} q \gamma^a \frac{i}{2} \gamma^5 \psi_q$, ...
and the axial-vector one, i.e., \( J_A^a = \bar{\psi}_q \gamma_\mu \gamma_5 \frac{\partial}{\partial \psi_q} \). Here, \( \tau^a \) with \( a = 1, 2, 3 \) is the \( SU(2) \) generator, i.e., the isospin Pauli matrices in which the repeated index \( a \) is summed up, and \( \gamma_5 \) is the Dirac gamma matrix \( (\gamma_5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3) \). The space-time-independent parameters \( \theta^0_{\psi,A} \) correspond to the rotation angle in the isospin space. In the QCD vacuum, however, the Lagrangian density is not invariant under the chiral transformation, and the quark condensates \( \langle \bar{\psi}_q \psi_q \rangle \) are finite. Actually, in the Lagrangian density, the scalar \( \bar{\psi}_q \psi_q \) and the pseudoscalar \( \bar{\psi}_q i\gamma_5 \tau^a \psi_q \) are not invariant under the \( SU_A(2) \) transformation: \( \bar{\psi}_q \psi_q \rightarrow \bar{\psi}_q \psi_q \cos \theta + \bar{\psi}_q i\gamma_5 \tau^a \psi_q \sin \theta \), \( \bar{\psi}_q i\gamma_5 \tau^a \psi_q \rightarrow -\bar{\psi}_q \psi_q \hat{\theta}^a \sin \theta + \bar{\psi}_q i\gamma_5 \tau^a \psi_q \cos \theta \). Here, the unit vector of the rotation angle \( \hat{\theta}^a \) is defined as \( \hat{\theta}^a = \theta^a / \theta \) with \( \theta = |\theta^a| \). Consequently, the vacuum have not the chiral symmetry, while the Lagrangian of the system is chiral invariant. This feature is called the spontaneous chiral symmetry breaking. When the chiral symmetry breaking occurs, Nambu-Goldstone modes such as pions appear. On the other hand, Higgs modes represent sigma mesons as the pion’s chiral partner. Fig. 1-1 shows the wine-bottle potential, also called the mexican-hat type potential, illustrating spontaneous chiral symmetry breaking for reference.

In Fig. 1-1, the center figure shows the symmetry breaking, while the potential figure on the right-hand side is symmetric. The chiral condensate selects a direction and rolls to a certain point in the potential on the right-hand side figure, so that the symmetry of the ground state is spontaneously broken. There are two fluctuations in the potential on the left-hand side figure which is the wine-bottle potential or mexican-hat type one. In the chiral model, one is Higgs modes called sigma modes in the direction of mountain, the other is Nambu-Goldstone modes called pionic modes\(^4\) in the direction of valley. Here, the chiral condensate moves in the valley-bottom without energy cost. These gapless modes are also called zero modes. In the coordinate axes, \( F(\phi) \) denotes

\(^4\) Actually, pions have a slightly smaller mass, so that the pion modes is often called the pseudo Nambu-Goldstone modes.
the free energy and \((\phi_1, \phi_2)\) represent order parameters.

In terms of the bosonized meson fields, \(\bar{\psi}_q \psi_q\) and \(\bar{\psi}_q i\gamma_5 \tau^a \psi_q\) correspond to the sigma meson field \(\sigma\) and the pion field \(\pi\), respectively. Here, \(\sigma\) and \(\pi^a\) are transformed as \(\sigma \rightarrow \sigma + \theta^A_\sigma \times \pi^a\) in the infinitesimal \(SU_A(2)\) transformation with \(\bar{\psi}_q \psi_q = \sigma\) and \(\bar{\psi}_q i\gamma_5 \tau^a \psi_q = \pi^a\). On the other hand, in the infinitesimal \(SU_V(2)\) transformation, \(\sigma\) and \(\pi^a\) are transformed as \(\sigma \rightarrow \sigma - \theta^\pi_\sigma \pi^a\) and \(\pi^a \rightarrow \pi^a + \theta^\pi_\sigma \sigma\). Thus, \(\sigma\) and \(\pi^a\) are not invariant under the \(SU(2)_L \otimes SU(2)_R\)-chiral transformation. However, the combination \(\sigma^2 + \pi^2\) is invariant under the chiral transformations: \(\sigma^2 + \pi^2 \rightarrow \sigma^2 + \pi^2\). Therefore, we can construct a possible chiral \(SU(2)_L \otimes SU(2)_R\)-invariant Lagrangian with the combination \(\sigma^2 + \pi^2\) which is a Lorenz-scalar. Examples of the \(SU(2)_L \otimes SU(2)_R\)-invariant models are the linear sigma model [13] or the Nambu-Jona-Lasinio (NJL) model [14].

### 1.1.2 QCD phase diagram

As previously mentioned, the running coupling constant in QCD decreases with increasing the energy scale. Thus, it is believed that the quark-hadron many-body system governed by QCD at high energy densities undergoes a phase transition from the chiral symmetric confined state, which corresponds to the hadronic matter, to the chiral symmetry breaking and deconfined state known as the quark-gluon plasma (QGP). This phase transition may occur under extreme conditions at high temperature and/or high density. From the fact that \(\Lambda_{QCD} \approx 200\) MeV, one finds that the critical temperature and baryon number density are around 200 MeV and 1 fm\(^{-3}\), respectively. The early universe right after the Big Bang corresponds to the high-temperature environment exceeding 10\(^{12}\)K and it would have experienced the QCD phase transition. On the other hand, the high-density environment exceeding 10\(^{12}\) kg/cc would be relevant to the interior-environment of compact stars such as neutron stars in which it is predicted that the ground state would form a condensate of quark-quark Cooper pairs, i.e., a diquark condensate, and a phenomenon called the color superconductivity would occur [13]. Here, the color superconducting phase has a rich structure since quarks, unlike electrons, have the color and flavor degrees of freedom as well as spin degrees of freedom, so that many different patterns of pairing are possible. Thus, in these extremely hot and/or dense environment, for strongly interacting many-body systems, there may exist various possible phases with characteristic dynamical symmetries and their rich symmetry breaking patterns such as hadronic or quark-gluonic phase, chiral symmetric or broken phase, color-flavor locked (CFL) or two-flavor color superconducting (2CS, uCS, dCS) phase and so forth [16]. For a recent review, see e.g., Ref. [17]. In Fig 1-2, the schematic QCD phase diagram in the temperature-baryon chemical potential \((T-\mu_B)\) plane is shown. The temperature and the baryon chemical potential are adopted as important external parameters for QCD in equilibrium. Here, the baryonic chemical potential \(\mu_B\) is a measure of the baryon number density \(\rho_B\).

Experimentally, the high-energy and high-intensity heavy-ion collisions provide the means of creating the hot and dense QCD matter [18]. In the year 2005, with the aim of reproduction the early universe condition, the Relativistic Heavy-ion Collider (RHIC) at Brookhaven National Laboratory (BNL) have been successful in creating the strongly coupled QGP (sQGP) and observing the
perfect liquid-like behavior, in which Au-Au (Gold+Gold nuclei) collisions with a center-of-mass energy per nucleon pair $\sqrt{s_{NN}}$, being 200 GeV, have been carried out as a typical example. Moreover, to investigate the QGP properties, further experiments with higher energies than RHIC have been performed by Large Hadron Collider (LHC) in European Organization for Nuclear Research (CERN), and from the year 2010 to the present some runs have been conducted, e.g., the Pb-Pb (Lead+Lead nuclei) collisions with $\sqrt{s_{NN}} = 2.76$ TeV, and so forth. Here, the RHIC/LHC-energy experiments cover the high-temperature and low-density regime. If we widely explore the QCD phase diagram, it is necessary to perform the beam energy scan (BES) with a wide collision energy which covers a wide range of baryon chemical potential $\mu$. In the near future, the experiments with the aim of looking for the dense QCD matter are planned at the Alternating Gradient Synchrotron (AGS) in RHIC, the Super Proton Synchrotron (SPS) in CERN and also at the future facilities with a high luminosity such as the Facility for Antiproton and Ion Research (FAIR) in GSI Helmholtz Centre for Heavy Ion Research (GSI), the Japan Proton Accelerator Research Complex (J-PARC) in a joint project between Japan Atomic Energy Agency (JAEA) and High Energy Accelerator Research Organization (KEK), and the Nuclotron-based Ion Collider Facility (NICA) in Joint Institute for Nuclear Research (JINR).

Theoretically, there are several ways to study the QCD phase diagram. While the most straightforward approach is the perturbative QCD (pQCD), the perturbative calculations only work at asymptotically high temperatures and chemical potentials in which the QCD coupling becomes small. Another method which is in principle very reliable is the lattice QCD (LQCD) simulation known as a first-principle calculation of the strong interaction [20]. By using this method, it is possible to verify the extremely high-temperature regime which is reproduced experimentally by RHIC/LHC. However, there is a severe problem that the LQCD calculation (Monte Carlo simulation) is not feasible at finite quark chemical potential $\mu_q$. Due to this problem which is called the sign problem, it is difficult to treat the finite density QCD. Thus, the current understanding of the QCD phase diagram is limited to the regime of high-$T$ and small-$\mu_q$, i.e., $\mu_q/T \ll 1$, so
that the phase structure of finite density QCD is still unclear. As a possible alternative approach to finite density systems, several effective models based on QCD can be useful tools. Until now, by using various effective models, the QCD phase structure has been often investigated at finite temperature and density. However, it is difficult to determine a phase boundary between the hadronic and quark-gluonic matters due to the quark confinement on the hadron side. Therefore, in the quark-hadron phase transition, no definite results have been obtained. Recently, an effective model which includes Polyakov loops as an approximate order parameter for the deconfinement phase transition, the so-called Polyakov-Nambu-Jona-Lasinio (PNJL) model [22], is well-known. In this thesis, however, the quark-hadron phase transition at finite temperature and baryon chemical potential is investigated in a different manner, in which the realized phase at given baryon density is determined by comparing the pressures of the hadronic matter and quark-gluonic matter.

There are some qualitatively similarities between the electron systems with Coulomb interactions and the quark-hadron systems with strong interactions, in despite of different material groups. For instance, the phase diagram of hole-doped cuprates with perovskite crystal structure shares similarity with the QCD phase diagram in terms of the strong correlativity, the competition between different order parameters, and the corresponding phase transitions. For the cuprates, the antiferromagnetic Neel phase and the high-temperature superconducting phase correspond to the chiral symmetry breaking phase and the color superconducting phase in QCD, respectively. In addition, the low-energy excitations of QCD, e.g., pions, bear a resemblance to that of electron spin systems, magnons. In condense matter physics, there is the SO(5) theory [23] described by the superspin which is a 5-component spin vector degrees of freedom with both the 3-component staggered magnetization and the 2-component Cooper pair condensate. Not only this theory unifies dynamically the SO(3) spin rotational symmetry and the U(1) gauge symmetry of electromagnetism, but also gives some qualitative insight to the phase structure of QCD. Thus, there is also an attempt to unify the chiral symmetry breaking and the color superconductivity in QCD, i.e., the SO(10) theory, as well as the SO(5) theory which unifies antiferromagnetism and superconductivity. In this way, unraveling the properties of quark-hadron matter links to the complementary understanding of the properties of strongly correlated electron systems.

1.2 Nambu-Jona-Lasinio model

The NJL model [14] provides a wide range of information for hadronic systems based on the concepts of chiral symmetry and its dynamical breaking. For some reviews, see e.g., Refs. [25] [26] [27] [28]. Originally, the NJL model was born in terms of nucleonic or hadronic degrees of freedom in order to describe the pion as a bound-state of a nucleon and an antinucleon, i.e., a nucleon-antinucleon Goldstone excitation. Nowadays, the NJL model is often used as a theory with quark degrees of freedom. In this section, we briefly introduce the original NJL model, which at that time

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5 To deal with finite chemical potential systems, we follow the treatment of Ref. [21] in this thesis.

6 In the SO(5) theory, it is argued that the SU(2)_L ⊗ SU(2)_R ⊗ U(1)_B-chiral symmetry and the SU(3)_c color gauge symmetry may get unified to the SO(10) theory [21].
was a model for nucleons, and the modern form of the NJL model as a model for quarks in which we leave out the explanation of the meson properties, for details, see above reviews. Also, an overview of the extended NJL model with a reasonable saturation property of nuclear matter are outlined.

1.2.1 Original NJL model

In 1961, before the discovery of quarks, the NJL model was formulated as a model for interacting nucleons [14]. This model is defined by the following Lagrangian density with massless nucleons:

\[ \mathcal{L}_{N}^{NJL} = \bar{\psi}_N i \gamma^\mu \partial_\mu \psi_N + G_s^N \left( [\bar{\psi}_N \psi_N]^2 + (\bar{\psi}_N i \gamma_5 \tau \psi_N)^2 \right). \]  

(1.3)

Here, \( \psi_N \) represents the \( SU(2) \) doublet describing the nucleons, i.e., nucleon fields, and \( \tau \) denote the isospin Pauli matrices. The nucleons interact through a four-point fermion interaction, which is a local interaction between four fermionic fields at a point, with the coupling constant \( G_s \) (see Fig. 1.3). Under the \( SU(2)_L \otimes SU(2)_R \)-chiral transformation, the above Lagrangian (1.3) is invariant, that is, chirally symmetric. When the coupling \( G_s \) is strong enough, the nucleons form nucleon-antinucleon pairs in the same way as the electrons form Cooper pairs in the microscopic theory of superconductivity, i.e., Bardeen-Cooper-Schrieffer (BCS) theory [21], and the non-zero condensates of nucleon-antinucleon pairs, \( \langle \bar{\psi}_N \psi_N \rangle \), are realized in which the chiral symmetry breaking occurs. Here, according to the Goldstone theorem [31], there should appear three Nambu-Goldstone bosons. The pion mass is significantly smaller than the nucleon mass, so that it was regarded as a massless boson. Thus, the pion at that time were interpreted as a Goldstone boson. By this interpretation, the current-algebra results could be explained. For this reason, the pion was described as a Goldstone boson of the spontaneously broken chiral symmetry with the nucleon as a fundamental particle. In fact, however, the pion is not a massless boson since it has a light mass. The smallness of the pion mass comes from the explicit symmetry breaking by the presence of a small bare nucleon mass in the Lagrangian. In this model, the interaction yields large nucleon self-energies even in the case of zero bare mass. This is seen as an explanation of the large nucleon mass, while being massless at the Lagrange level. After the establishment of QCD, the NJL model is reinterpreted as a quark model, which is a low-energy effective theory with quark degrees of freedom, so that it is also called the quark NJL (qNJL) model.
1.2.2 Quark NJL model

In the new interpretation of the NJL model, a vacuum is described by a quark-antiquark condensate, not a nucleon-antinucleon one, in which a pion is still interpreted as a Goldstone boson. Although the model does not contain gluons, it is a successful low energy approximation to QCD in which form factors and other finite-range effects are ignored, since the symmetry of the model reflects that of the strong interaction. As a result, this model provides that hadron masses are generated by the spontaneous symmetry breaking of the vacuum. Here, the model is nonrenormalizable, so that the results are scheme-dependent. Also, the quark confinement is not taken into account in this model. For two quark flavors, the NJL Lagrangian density with massless quarks interacting through zero-range interactions can be written as

$$\mathcal{L}_{N}^{NJL} = \overline{\psi}_q i\gamma^\mu \partial_\mu \psi_q + G_s^2 \left( (\overline{\psi}_q \psi_q)^2 + (\overline{\psi}_q \gamma_5 \tau \psi_q)^2 \right).$$  

(1-4)

Here, $\psi_q$ represents the quark field and $G_s^2$ denotes the coupling constant for the scalar-isoscalar four-point interaction.

In the mean field approximation (MFA) which is an approximation simplifies calculations drastically, an effective quark mass $m_q$ known as a constituent quark mass is obtained in which $m_q$ is derived by coupling to itself via the scalar and isoscalar channel. Under this approximation, bilinear quantities in the quark fields, $\overline{\psi}_q \Gamma \psi_q$, are replaced by $\langle \overline{\psi}_q \Gamma \psi_q \rangle = \langle \overline{\psi}_q \Gamma \psi_q \rangle - \langle \overline{\psi}_q \Gamma \psi_q \rangle$ where $\langle \overline{\psi}_q \Gamma \psi_q \rangle$ denotes the expectation value and $\langle \overline{\psi}_q \Gamma \psi_q \rangle$ represents the fluctuation around the displacement from the quantity $\overline{\psi}_q \Gamma \psi_q$ with $\Gamma = 1, \gamma_5, \gamma_\mu, \gamma_\mu \gamma_5$, etc. Also, it is assumed that fluctuations are small, so that the fluctuations are linearized. Then, the four-point interactions are approximated by $\langle \overline{\psi}_q \Gamma \psi_q \rangle^2 \sim -\langle \overline{\psi}_q \Gamma \psi_q \rangle^2 + 2 \overline{\psi}_q \Gamma \psi_q \langle \overline{\psi}_q \Gamma \psi_q \rangle^2$. Here, we consider two nonvanishing terms, the chiral condensate $\langle \overline{\psi}_q \gamma^\mu \psi_q \rangle \neq 0$ and the quark number density $\rho_q \equiv \langle \overline{\psi}_q \gamma_5 \psi_q \rangle \neq 0$. As a result, the mean-field Lagrangian density is obtained as

$$\mathcal{L}_{q}^{MF} = \overline{\psi}_q (i\gamma^\mu \partial_\mu - m_q) \psi_q - G_s^2 \langle \overline{\psi}_q \gamma^\mu \psi_q \rangle^2,$$  

(1-5)

where $m_q$ is a dynamical quark mass, and is defined as

$$m_q = -2G_s^2 \langle \overline{\psi}_q \gamma^\mu \psi_q \rangle.$$  

(1-6)

The vacuum expectation value is calculated as $\langle \overline{\psi}_q \gamma^\mu \psi_q \rangle = -i \int \frac{d^4p}{(2\pi)^4} \text{Tr}(S(p))$ where $S(p) = 1/(\slashed{p} - m + i\epsilon)$, in which the trace is to be taken in flavor, color and Dirac space. Here, the integral is divergent, so that the regularization is required. When imposing a cutoff on $p^2 \leq A_q^2$ after the $p_0$-integration, the integral can be solved analytically.

To deal with finite density systems, we introduce the chemical potential $\mu_q$ into the operator $\mathcal{H} = \mathcal{H}_M - \mu_q \rho_q$ where $\mathcal{H}_M$ is the mean-field Hamilton density defined as $\mathcal{H}_M = -i \overline{\psi}_q \gamma^\mu \nabla \psi_q + m_q \overline{\psi}_q \psi_q + \mu_q \overline{\psi}_q \gamma^0 \psi_q + G_s^2 \langle \overline{\psi}_q \gamma^\mu \psi_q \rangle$. Under this Hamiltonian density, we can calculate physical quantities. Here, the expectation value of the chiral condensate can also be written as

$$\langle \overline{\psi}_q \gamma^\mu \psi_q \rangle = - \int \frac{d^4p}{i(2\pi)^4} \text{Tr}(iS_q(p)) = -4N_f^2 N_q^2 \int \frac{d^4p}{i(2\pi)^4} \frac{1}{p^2 - m_q^2 - i\epsilon},$$  

(1-7)

\(^5\)The condensate $\langle \overline{\psi}_q \gamma_5 \tau \psi_q \rangle$ is assumed to be zero since it could break parity, and thus $\langle \overline{\psi}_q \gamma_5 \tau \psi_q \rangle^2 \sim 0$.

\(^6\)In the three-momentum noncovariant cutoff scheme, $\int_0^{\Lambda_q} \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{p^2 + m^2}} = \Lambda_q \sqrt{m^2 + \Lambda_q^2} - m^2 \sinh^{-1}(\Lambda_q^2/m^2)/4\pi^2$. 

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where \( iS_q(p) = (p^\mu \gamma_\mu - m_q - i\epsilon)^{-1} \) is the quark propagator in which \( p_0 \) is replaced into \( p_0 + \mu_q \), and \( N_c^q \) and \( N_f^q \) are the numbers of color and flavor, respectively. Further, by using the imaginary-time formalism or Matsubara formalism [22], we can extend the system under consideration to finite temperature systems, in which the time component of four-momentum after Wick rotation, \( p_4 \), and the \( p_0 \)-integral \( \int dp_0/(2\pi) \) are respectively replaced by Matsubara frequency \( \omega_n = (2n + 1)\pi T \) and Matsubara sum \( iT \sum_{n=-\infty}^{\infty} \), i.e.,

\[
\int \frac{d^3p}{i(2\pi)^4} f(p_0, p) \longrightarrow T \sum_{n=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3} f(i\omega_n + \mu_q, p), \tag{1-8}
\]

where \( n, \mu_q \) and \( T = 1/\beta \) represent an integer, the quark chemical potential and temperature, respectively. Consequently, the expectation value of the chiral condensate at finite density and temperature is written as

\[
\langle \overline{\psi}_q \psi_q \rangle = \nu_q \int \frac{d^3p}{(2\pi)^3} \frac{m_q}{\sqrt{p^2 + m_q^2}} (n_+^q - n_-^q), \tag{1-9}
\]

where \( \nu_q = 2N_c^q N_f^q \) is the degeneracy factor and \( n_+^q \) are the quark number distribution functions defined as

\[
n_\pm^q = \left[ e^{\beta(\pm \sqrt{p^2 + m_q^2} - \mu_q)} + 1 \right]^{-1}, \tag{1-10}
\]

which are related to the total quark number density at finite temperature and chemical potential [22].

\[
\langle \overline{\psi}_q \gamma^0 \psi_q \rangle = \nu_q \int \frac{d^3p}{(2\pi)^3} (n_+^q + n_-^q - 1). \tag{1-11}
\]

Here, the contribution of the occupied negative energy states is eliminated from the quark number density in which we replace \( n_+^q \) as \( n_+^q - 1 \).

A self-consistent equation for the dynamical quark mass, i.e., \( m_q = -2G^q_\sigma \langle \overline{\psi}_q \psi_q \rangle \) with Eqs. (1-9) and (1-10), is the so-called gap equation in BCS theory. The gap equation might have more than one solution, so that a criterion is required to decide which solution is the correct one. A possible way is to take a procedure in statistical physics, that is, the minimization of the thermodynamic potential [22]. The thermodynamic potential density is defined as

\[
\omega_q = \langle \mathcal{H}^{MF}_q \rangle - \mu_q \langle N_q \rangle - \frac{1}{\beta} \langle S_q \rangle, \tag{1-12}
\]

where

\[
\langle \mathcal{H}^{MF}_q \rangle = \langle \overline{\psi}_q (\gamma \cdot p) \psi_q \rangle - G^q_\sigma \langle \overline{\psi}_q \psi_q \rangle^2, \tag{1-13}
\]

\[
\langle N_q \rangle = \langle \overline{\psi}_q \gamma^0 \psi_q \rangle, \tag{1-14}
\]

\[
\langle S_q \rangle = -\nu_q \int \frac{d^3p}{(2\pi)^3} \left[ n_+^q \ln n_+^q + (1 - n_+^q) \ln(1 - n_+^q) + n_-^q \ln n_-^q + (1 - n_-^q) \ln(1 - n_-^q) \right]. \tag{1-15}
\]

\[\text{At zero temperature, } \rho_q = \langle \overline{\psi}_q \gamma^0 \psi_q \rangle = -\int \frac{d^3p}{(2\pi)^3} \text{Tr}(\gamma_0 iS_q(p)) = -4N_c^q N_f^q \int \frac{d^3p}{(2\pi)^3} \frac{\delta^3(p)}{p^2 - m_q^2 - i\epsilon}. \]

\[\text{It is appropriate to use the thermodynamic potential since the temperature and the chemical potential are fixed and the particle number can vary.}\]
Here, the thermodynamical average\(^{11}\) such as \(\langle \bar{\psi}_q \psi_q \rangle\) or \(\langle \bar{\psi}_q \gamma^0 \psi_q \rangle\) have the same forms as Eqs. (1.19) and (1.11), in which \(\langle \bar{\psi}_q (\gamma \cdot p) \psi_q \rangle\) is expressed as
\[
\langle \bar{\psi}_q (\gamma \cdot p) \psi_q \rangle = \nu_q \int \frac{d^3 p}{(2\pi)^3} \frac{p^2}{\sqrt{p^2 + m_q^2}} (n^+_q - n^-_q) . \tag{1.16}
\]
By minimizing \(\omega_q\) with respect to \(m_q\), we get the following equation:
\[
\frac{\partial \omega_q}{\partial m_q} = \nu_q \frac{p^2}{(p^2 + m_q^2)^{3/2}} (n^+_q - n^-_q) \left[ -m_q - 2G^q \langle \bar{\psi}_q \psi_q \rangle + 2G^{\gamma \psi}_q \langle \bar{\psi}_q \gamma^0 \psi_q \rangle \right] = 0 , \tag{1.17}
\]
which leads to the gap equation. Thus, the criterion for the correct solution of the gap equation is obtained, in which although the gap solutions are the extreme values of \(\omega_q\), the stable solution is the only one solution which corresponds to the global minimum of \(\omega_q\). Here, the thermodynamic quantities can be derived from \(\omega_q\). For instance, the pressure is given by \(p_q(T, \mu_q) = -\omega_q(T, \mu_q)\) with the normalization \(p_q(0, m_q(T=0)) = 0\) where \(\mu_q = m_q(T=0)\) leads to \(\rho_q = 0\). Therefore, the stable solution also corresponds to the solution which leads to the largest pressure. In this thesis, we will calculate the pressure \(p_q\) in order to determine the stable gap solution.

Below, by adding the term \(\mathcal{L}'_q = -G^q_\gamma (\bar{\psi}_q \gamma^\mu \psi_q)\) to the Lagrangian (1.14), we consider the case including vector interaction, which is chirally invariant and naturally appears in a QCD framework, in which for simply we assume the vector channel to be flavor-independent. Consequently, the mean-field Lagrangian density (1.15) is recast into
\[
\mathcal{L}'^{\text{MF}}_q = \bar{\psi}_q (i\gamma^\mu \partial_\mu - m_q) \psi_q - 2G^q_\gamma \langle \bar{\psi}_q \gamma^0 \psi_q \rangle \psi_q - (G^q_\gamma \langle \bar{\psi}_q \gamma^0 \psi_q \rangle)^2 + G^q_\gamma \langle \bar{\psi}_q \gamma^0 \psi_q \rangle)^2 \tag{1.18}
\]
where the expectation values are extended to the finite temperature case. The mean-field Hamilton density is also recast as
\[
\mathcal{H}'^{\text{MF}} = -i\bar{\psi}_q \gamma \cdot \nabla \psi_q + m_q \bar{\psi}_q \psi_q + 2G^q_\gamma \langle \bar{\psi}_q \gamma^0 \psi_q \rangle \bar{\psi}_q \gamma^0 \psi_q + (G^q_\gamma \langle \bar{\psi}_q \gamma^0 \psi_q \rangle)^2 - G^q_\gamma \langle \bar{\psi}_q \gamma^0 \psi_q \rangle)^2 \tag{1.19}
\]
As a result of introducing the chemical potential as
\[
\mathcal{H}' = \mathcal{H}'^{\text{MF}} - \mu_q \langle \psi_q \rangle \langle \bar{\psi}_q \rangle,
\]
where \(\mu_q\) is the renormalized chemical potential define as
\[
\mu_q' = \mu_q - \bar{\mu}_q = \mu_q - 2G^q_\gamma \langle \bar{\psi}_q \gamma^0 \psi_q \rangle , \tag{1.20}
\]
the thermodynamic potential (1.12) is also rewritten, in which \(\langle \mathcal{H}'^{\text{MF}} \rangle = \langle \bar{\psi}_q (\gamma \cdot p) \psi_q \rangle - G^q_\gamma \langle \bar{\psi}_q \gamma^0 \psi_q \rangle^2 + G^q_\gamma \langle \bar{\psi}_q \gamma^0 \psi_q \rangle)^2\) and \(n^\pm_q = [e^{\beta (p^2 + m^2_q - n^\pm_q)} + 1]^{-1}\). Here, the quark number distribution functions \(n^\pm_q\) are obtained again by minimizing \(\omega_q\) with respect to \(n^\pm_q\):
\[
\frac{\partial \omega_q}{\partial n^\pm_q} = \nu_q \left[ \pm \frac{1}{\sqrt{P^2 + m^2_q}} (p^2 - 2m_q G^q_\gamma \langle \bar{\psi}_q \psi_q \rangle - (\mu_q - 2G^q_\gamma \langle \bar{\psi}_q \gamma^0 \psi_q \rangle) + \frac{1}{\beta} \ln \frac{n^+_q}{n^-_q} \right] = 0 . \tag{1.22}
\]
\(^{11}\) An arbitrary thermodynamical average is given by \(\langle O \rangle = \text{Tr} O e^{-\beta (\mathcal{H}_\text{MF} - \mu_q N_q)}/\text{Tr} e^{-\beta (\mathcal{H}_\text{MF} - \mu_q N_q)}\).
\(^{12}\) At zero temperature, \(\langle \bar{\psi}_q (\gamma \cdot p) \psi_q \rangle = \int d^3 p / (2\pi)^3 \text{Tr} \gamma \cdot p S_q(p) = -4N^q_q g_f \int \frac{d^3 p}{(2\pi)^3} \frac{p^2}{p^2 - m^2_q + i\epsilon} .\)
1.2.3 Nucleon NJL model

When describing the nuclear matter as a relativistic system, it is necessary to reproduce the observed saturation properties of nuclear matter. The most prominent model is the Walecka model [33] which has succeeded in describing phenomenologically the saturation curve of symmetric nuclear matter as the result of the cancellation of a large scalar and a large vector potential, which are respectively derived by the $\sigma$-meson and $\omega$-meson exchanges between nucleons. However, this model does not have chiral symmetry, while it has been rather successful in explaining important properties of nuclei and nuclear matter.

With the aim of constructing a saturating model for nuclear matter based on purely fermions, the NJL model for nucleon degrees of freedom, which is called the nucleon NJL (nNJL) model, has been applied in which an additional scalar-vector interaction term, which effectively make the scalar coupling constant density-dependent, is introduced in order to reproduce the nuclear matter saturation properties [34]. In the extended NJL (ENJL) model with scalar-vector interaction [34, 35, 36, 37, 38], the Lagrangian density of the massless $SU(2)_L \otimes SU(2)_R$ case is given by

$$\mathcal{L}^\text{Ext}_N = \bar{\psi}_N i \gamma^\mu \partial_\mu \psi_N + G^N_{4_s} \left[ (\bar{\psi}_N \gamma^5 \psi_N)^2 - (\bar{\psi}_N \gamma^5 \tau^5 \psi_N)^2 \right]$$

$$- G^N_{4_v} \left[ (\bar{\psi}_N \gamma^\mu \psi_N)^2 + (\bar{\psi}_N \gamma^5 \tau^5 \gamma^\mu \psi_N)^2 \right]$$

$$+ G^N_{8sv} \left[ (\bar{\psi}_N \gamma^5 \psi_N)^2 - (\bar{\psi}_N \gamma^5 \tau^5 \psi_N)^2 \right] \left[ (\bar{\psi}_N \gamma^\mu \psi_N)^2 - (\bar{\psi}_N \gamma^5 \tau^5 \gamma^\mu \psi_N)^2 \right],$$

(1.23)

where the last term with $G^N_{8sv}$ is important to simultaneously obtain both a realistic vacuum mass and a saturation, in which the nucleon field $\psi_N$ is treated as a fundamental field, not as a composite one. In this context, the problem of matter stability within the NJL model is reinvestigated [39] in which it is argued that bound nucleonic matter with spontaneously broken chiral symmetry is not possible within the original NJL model. Incidentally, for the stability of quark droplets, a similar result is obtained in the NJL model with the time-dependent Hartree-Fock (TDHF) formalism [40]. Therefore, the ENJL model with scalar-vector interaction is useful in dealing with the nuclear matter since the bulk static properties such as saturation and binding energy of nuclear matter is well reproduced. This treatment corresponds to the Walecka model [33]. Some of similar models describe a nucleon as a quark-di-quark bound-state [41] in which the quark substructure produces a nuclear matter saturation mechanism which plays the same role as the scalar-vector interaction term. Here, this substructure may be provide a possible microscopic foundation for the phenomenological approach with scalar-vector interactions, while the scalar-vector coupling term accounts for the density-dependence of the scalar coupling between nucleons, that is a manifestation of the nucleon’s composite nature.

In this thesis, when dealing with the nuclear matter, we start with the following Lagrangian

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13Although the Walecka model has a stiff equation of state, the ENJL model with scalar-vector interaction as a model of the nuclear matter has a rather reasonable saturation properties due to the effect of the additional scalar-vector interaction.
with a reasonable saturation properties of nuclear matter \[ 35, 39]:
\[
\mathcal{L}_N^{\text{ext}} = \bar{\psi}_N i\gamma^\mu \partial_\mu \psi_N + G_N^S \left[ (\bar{\psi}_N \psi_N)^2 - (\bar{\psi}_N i\gamma_5 \tau \psi_N)^2 \right] - G_N^S (\bar{\psi}_N \gamma^\mu \psi_N)^2 - G_S^N \left[ (\bar{\psi}_N \psi_N)^2 + (\bar{\psi}_N i\gamma_5 \tau \psi_N)^2 \right] (\bar{\psi}_N \gamma^\mu \psi_N)^2.
\]  
(1.24)

Here, the third term with \( G_N^S \) reflects a chiral invariant short-range repulsion between nucleons and the last term with \( G_N^S \) leads to an effective density-dependent coupling constant, \( G_N^S \rightarrow G_N^S (\rho_N) = G_S^N (1 - G_S^N / G_S^N \cdot \rho_N^2) \), which pushes the chiral symmetry restoration point to the high-density side, and makes an incompressibility lower.

In a QCD-inspired many-body model for the nucleus, where the strong coupling regime is controlled by a three-body string-type force and the weak coupling regime is dominated by a pairing force, one finds that the effective model becomes a NJL-type model and leads to that the chiral fields are identified with the two-particle strings, which are natural in a QCD framework, in which the nuclear matter saturation is also derived \[ 14] .

1.3 Main objective and thesis organization

In this chapter, we have given a brief introduction to two remarkable features of QCD and the exploration of the QCD phase diagram. Also, we have mentioned the NJL model as a quark model and also as a model for nuclear matter with a reasonable saturation property. In this section, we explain the research objective and the organization of this thesis.

In this thesis, the main objective is to investigate the quark-hadron phase transition and draw a phase diagram between the symmetric nuclear matter and quark matter without diquark correlation. In Ref. \[ 12 \], it is argued that a first-order quark-hadron phase transition is obtained at zero temperature and the quark-hadron phase transition occurs after the chiral symmetry restoration in nuclear matter. Thus, we extend the zero-temperature results in Ref. \[ 12 \] to the finite temperature and chemical potential case in which for the nuclear matter we adopt the extended NJL model with scalar-vector eight-point interactions and treat a nucleon field as a fundamental fermion field with the number of colors \( N_c \), being one, while for the quark matter we adopt the extended NJL model including scalar-vector eight-point interaction with \( N_c = 3 \). This thesis is based on the work of Ref. \[ 13 \].

The rest of this thesis is organized as follows. In the next chapter, we introduce the extended NJL model at finite temperature and baryon chemical potential for nuclear and quark matters within the mean field approximation. The results obtained in this model are identical to those derived by minimizing of thermodynamic potential density. Also, we present a method for determining the model-parameters of quark matter and that of nuclear matter, respectively. In Chapter 3 the numerical results such as the gap solutions with the parameters determined at zero temperature are given, in which the realized phase is determined by comparing the pressure of each phase. Here, we present only a typical case for the free parameter. Thus, another setting results are outlined in Appendix A. For more details on numerical results, see Appendix B. The chiral and quark-hadron
phase transitions at finite temperature and density are described in this model. In Chapter 4 the phase diagram with scalar-vector eight-point interaction at finite temperature and baryon chemical potential is presented. Moreover, the dependence of the scalar-vector coupling constant on the phase diagram is shown. The last section is devoted to a summary and concluding remarks. In Appendix C and D we list some of the subjects not covered in this thesis.
Chapter 2

Extended NJL Model for Nuclear and Quark Matters

2.1 Extended NJL model for nuclear and quark matters at finite temperature and density

In considering the nuclear matter, from the fact that a four-point interaction term, which is a characteristic of the NJL Lagrangian, effectively comes out of a string model approach [41], it is possible to consider a NJL-type model with the four-point interaction as one of the possible models of nuclear matter. In addition, it has been observed that the nuclear saturation property is well reproduced by introducing the scalar-vector and isoscalar-vector eight-point interaction in the original NJL model in which the nucleon being then a fundamental fermion, and hence we adopt here an extended NJL model with rather reasonable nuclear saturation properties as a model for nuclear matter. Although this NJL-type model for the nucleon contains a conceptual problem in which an artificial Goldstone mode appears, we regard a meson, as well as a nucleon, as a fundamental particle and do not treat a pion-like excitation here with the same treatment as in some works [31, 33, 36, 37, 38].

As for quark matter, we also adopt the extended NJL model including the scalar-vector eight-point interaction with $N_c = 3$ in terms of quark degrees of freedom. Here, we consider free quark phase without quark-pair correlations. Namely, we take into account the color non-superconducting quark matter. On the other hand, for nuclear matter, we concentrate on the isospin-symmetric nuclear matter. Thus, as the first step to investigate the quark-hadron phase transition, we deal with the symmetric nuclear matter and free quark matter in this thesis. Of course, the color-superconducting phase may exist at finite density, and it is believed that neutron star matter undergoes a phase transition to quark matter at high density. The investigation of the above-mentioned case is left for the future as a possible next challenging task.
### 2.1.1 Lagrangian density, gap equation, and pressure

In this section, following Ref. [12][33], the same NJL-type models with the scalar-vector interaction are given for nuclear and quark matters at finite temperature and density, in which the model parameters and the number of colors are different. Let us start with the following Lagrangian density for nuclear and quark matters:

\[
\mathcal{L}_i = \bar{\psi}_i i\gamma^\mu \partial_{\mu} \psi_i + G^i_s \left[ (\bar{\psi}_i \psi_i)^2 + (\bar{\psi}_i i\gamma_5 \tau \psi_i)^2 \right] - G^i_v (\bar{\psi}_i \gamma^\mu \psi_i) (\bar{\psi}_i \gamma_\mu \psi_i) - G^i_{sv} \left[ (\bar{\psi}_i \psi_i)^2 + (\bar{\psi}_i i\gamma_5 \tau \psi_i)^2 \right] (\bar{\psi}_i \gamma^\mu \psi_i) (\bar{\psi}_i \gamma_\mu \psi_i) ,
\]

(2.1)

where the subscript/superscript \(i\) denotes nothing but an index that represents the case of nuclear matter (\(i = N\)) or quark matter (\(i = q\)). Here, \(\psi\) represents fermion field, that is, \(\psi_N\) is the nucleon field and \(\psi_q\) is the quark field. The first two terms are the original NJL model Lagrangian density. The third term is a vector-vector repulsive term with \(G^i_v\). The last term is a scalar-vector and isoscalar-vector coupling term with \(G^i_{sv}\). Parameters \(G^i_v\) and \(G^i_{sv}\) represent a coupling constant of four-point vector-vector interaction and that of eight-point scalar-vector interaction, respectively. As mentioned in the previous chapter, for nuclear matter, pure NJL interaction alone is not enough to reproduce the property of nuclear saturation in the original NJL model, and hence we introduce the last two terms in Eq. (2.1) so as to reproduce the nuclear matter saturation properties. Also, for quark matter, this term plays an important role in pushing the chiral restoration point to higher density side. Here, this model is nonrenormalizable, so that we adopt a three-momentum cutoff scheme in which the cutoff parameter \(\Lambda\) is introduced.

Under the mean field approximation\[1], the mean field Lagrangian density \(\mathcal{L}^{MF}_i\) and the mean field Hamiltonian density \(\mathcal{H}^{MF}_i\) for nuclear matter (\(i = N\)) and quark matter (\(i = q\)) are respectively obtained as

\[
\mathcal{L}^{MF}_i = \bar{\psi}_i (i\gamma^\mu \partial_{\mu} - m_i) \psi_i - \bar{\psi}_i \gamma^0 \psi_i + C_i ,
\]

\[
\mathcal{H}^{MF}_i = -i \bar{\psi}_i \gamma \cdot \nabla \psi_i + m_i \bar{\psi}_i \psi_i + \bar{\psi}_i \gamma^0 \psi_i - C_i ,
\]

\[
C_i \equiv -G^i_s (\bar{\psi}_i \psi_i)^2 + G^i_v (\bar{\psi}_i \gamma^0 \psi_i)^2 + 3G^i_{sv} (\bar{\psi}_i \psi_i)^2 (\bar{\psi}_i \gamma^0 \psi_i)^2 ,
\]

(2.2)

where

\[
m_i = -2 \left[ G^i_s - G^i_{sv} \langle \bar{\psi}_i \gamma^0 \psi_i \rangle \right] \langle \bar{\psi}_i \psi_i \rangle ,
\]

\[
\bar{\mu}_i = 2 \left[ G^i_v + G^i_{sv} \langle \bar{\psi}_i \psi_i \rangle \right] \langle \bar{\psi}_i \gamma^0 \psi_i \rangle .
\]

(2.3)

(2.4)

Here, the symbol \(\langle \cdots \rangle\) denotes the finite-temperature expectation value that represents the thermal average. In addition, we introduce the chemical potential \(\mu_i\) to deal with finite density systems:

\[
\mathcal{H}^{MF}_i = \mathcal{H}^{MF}_i - \mu_i \bar{\psi}_i \psi_i
\]

\[
- i \bar{\psi}_i \gamma \cdot \nabla \psi_i + m_i \bar{\psi}_i \psi_i - \mu_i \bar{\psi}_i \gamma^0 \psi_i - C_i ,
\]

(2.5)

---

1As may be seen in §1.2.2, we replace bilinear quantities \(\bar{\psi}_i \Gamma \psi_i\) with \(\langle \bar{\psi}_i \Gamma \psi_i \rangle\) in fermion fields, in which the fluctuation \(\delta (\bar{\psi}_i \Gamma \psi_i) = (\bar{\psi}_i \Gamma \psi_i - \langle \bar{\psi}_i \Gamma \psi_i \rangle)\) is linearized. As a result, we consider two nonvanishing terms: \(\langle \bar{\psi}_i \psi_i \rangle \neq 0\) and \(\langle \bar{\psi}_i \gamma^0 \psi_i \rangle \neq 0\). Here, \(\rho_i = \langle \bar{\psi}_i \psi_i \rangle = \langle \bar{\psi}_i \gamma^0 \psi_i \rangle\) represents the fermion number density.
where $\mu_i^r$ is the effective chemical potential:

$$\mu_i^r = \mu_i - \bar{\mu}_i = \mu_i - 2 \left[ G^{0} + G_{sv} \langle \overline{\psi}_i \psi_i \rangle \right] \langle \overline{\psi}_i \gamma^0 \psi_i \rangle . \tag{2.6}$$

Here, the expectation values at finite temperature is given as

$$\langle \overline{\psi}_i \psi_i \rangle = \nu_i \int \frac{d^3p}{(2\pi)^3} \frac{m_i}{\sqrt{p^2 + m_i^2}} (n_i^+ - n_i^-) , \tag{2.7}$$

$$\langle \overline{\psi}_i \gamma^0 \psi_i \rangle = \nu_i \int \frac{d^3p}{(2\pi)^3} (n_i^+ + n_i^- - 1) \tag{2.8}$$

with

$$\nu_i = 2 N^i_1 N^i_c , \tag{2.9}$$

$$n_i^\pm = \left[ e^{\beta (\pm \sqrt{p^2 + m_i^2} - \mu_i)} + 1 \right]^{-1} , \tag{2.10}$$

where $\nu_i$ is the degeneracy factor in which $N^i_1$ and $N^i_c$ represent the numbers of color and flavor, and $n_i^\pm$ are the fermion number distribution functions with $\beta = 1/T$, in which $T$ is temperature. Here, we have eliminated the contribution of the occupied negative energy states\footnote{Although the contribution of the occupied negative energy states are eliminated, namely, $n_i^0 \to n_i^1 - 1$ in Eq. (2.8), the negative energy states are taken into account by introducing the cutoff Lambda in this model, except for the number density in Eq. (2.8).} from the nucleon and/or quark number density itself in Eq. (2.8). As a result, we obtain a self-consistent set of equations, Eqs. (2.3) and (2.6)-(2.8) with (2.10), in which Eq. (2.3) is the gap equation.

The thermodynamic potential density $\omega_i$ for nuclear and quark matters are defined as

$$\omega_i = \langle \mathcal{H}_i^{MF} \rangle - \mu_i \langle N_i \rangle - \frac{1}{\beta} \langle S_i \rangle , \tag{2.11}$$

where

$$\langle \mathcal{H}_i^{MF} \rangle = \langle \overline{\psi}_i (\gamma \cdot \mathbf{p}) \psi_i \rangle - G^{0} \langle \overline{\psi}_i \psi_i \rangle^2$$

$$+ G^{0} \langle \overline{\psi}_i \gamma^0 \psi_i \rangle^2 + G_{sv} \langle \overline{\psi}_i \psi_i \rangle \langle \overline{\psi}_i \gamma^0 \psi_i \rangle , \tag{2.12}$$

$$\langle N_i \rangle = \langle \overline{\psi}_i \gamma^0 \psi_i \rangle = \rho_i , \tag{2.13}$$

$$\langle S_i \rangle = -\nu_i \int \frac{d^3p}{(2\pi)^3} \left[ n_i^+ \ln n_i^+ + (1 - n_i^+) \ln (1 - n_i^+) \right] , \tag{2.14}$$

and

$$\langle \overline{\psi}_i (\gamma \cdot \mathbf{p}) \psi_i \rangle = \nu_i \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{\sqrt{p^2 + m_i^2}} (n_i^+ - n_i^-) . \tag{2.15}$$

As mentioned in \cite{1}, by minimizing $\omega_i$ with respect to $m_i$ and $n_i^\pm$, we regain the gap equation in Eq. (2.3) and the fermion number distribution functions in Eq. (2.10). From the thermodynamic potential density $\omega_i$, the thermodynamic quantities such as the pressure $p_i$ can be derived. The pressures of nuclear matter ($i = N$) and quark matter ($i = q$) are given as

$$p_i = - \left[ \langle \mathcal{H}_i^{MF} \rangle (T, \mu_i) - \langle \mathcal{H}_i^{MF} \rangle (T = 0, \mu_i = m_i (T = 0)) \right] + \mu_i \langle N_i \rangle + \frac{1}{\beta} \langle S_i \rangle . \tag{2.16}$$
where the zero-density expectation value of the mean field Hamiltonian is subtracted in the vacuum.

This subtraction method is the same as that used in the expression for energy density, which is given later. We will discuss the determination of the realized phase by comparing the pressures of nuclear matter and quark matter.

### 2.1.2 Model parameters

For nuclear matter, the extended NJL model has four parameters: $G_s^N, G_v^N, G_{sv}^N$, and $\Lambda_N$. These parameters are determined by four conditions at zero temperature: the nucleon mass in vacuum, $m_N(\rho_N=0) = 939$ MeV, the normal nuclear density, $\rho_N^0 = 0.17$ fm$^{-3}$, the nucleon mass at normal nuclear density, $m_N(\rho_N^0) = 0.6m_N(\rho_N = 0)$, and the saturation properties of nuclear matter, $W_N(\rho_N^0) = -15$ MeV. At zero temperature, the fermion number distribution functions $n_i^\pm$ in Eq. (2.10) are reduced to the Heaviside step function: $n_i^+ = \theta(\mu_i^+ - \sqrt{p^2 + m_i^2})$ and $n_i^- = 1$. In the nuclear matter case, the gap equation at zero temperature is expressed as

$$m_N = -2G_s^N \left[ 1 - \frac{G_{sv}^N}{G_s^N} \rho_N^2 \right] \langle \overline{\psi}_N \psi_N \rangle,$$

(2.17)

where

$$\langle \overline{\psi}_N \psi_N \rangle = -\frac{\nu_N m_N}{2\pi^2} \int_{p_1^N}^{\Lambda_N} d|p| \frac{p^2}{\sqrt{p^2 + m_N^2}},$$

$$\rho_N(T=0) = \frac{\nu_N}{6\pi^2} n_F^N.$$

Here, $p_F^N = \left( \frac{\mu_N}{\rho_N} - m_N^2 \right)^{1/2}$ and $\mu_N = \mu_N - 2 \left( G_v^i + G_{sv}^i \langle \overline{\psi}_i \psi_i \rangle \right) \rho_N$ are the Fermi momentum and the effective chemical potential for nuclear matter, respectively, in which $\nu_N = 2N_F^N N_c^N$ with $N_F^N = 2$ and $N_c^N = 1$. The symbol $\langle \cdots \rangle$ denotes the expectation value at zero temperature. It is seen from Eq. (2.17) that the scalar-vector coupling term $G_{sv}^N$ indeed leads to a density-dependent coupling $G_{sv}^N(\rho_N) = G_{sv}^N, G_s^N \left[ 1 - \frac{G_{sv}^N}{G_s^N} \rho_N^2 \right]$.

The energy density per single nucleon at finite baryon density and zero temperature is evaluated as

$$W_N(\rho_N) = \frac{\langle \mathcal{H}_N^{MF}(\rho_N) - \langle \mathcal{H}_N^{MF}(\rho_N = 0) \rangle }{\rho_N} = m_N(\rho_N = 0),$$

(2.20)

where

$$\langle \mathcal{H}_N^{MF}(\rho_N) \rangle = \langle \overline{\psi}_N (\gamma \cdot p) \psi_N \rangle - G_s^N \left[ 1 - \frac{G_{sv}^N}{G_s^N} \rho_N^2 \right] \langle \overline{\psi}_N \psi_N \rangle^2 + G_v^N \rho_N^2$$

(2.21)

with

$$\langle \overline{\psi}_N (\gamma \cdot p) \psi_N \rangle = -\frac{\nu_N}{2\pi^2} \int_{p_1^N}^{\Lambda_N} d|p| |p|^4 \frac{|p|^2}{\sqrt{p^2 + m_N^2}}.$$ 

(2.22)

Here, in Eq. (2.20), the vacuum value for $\langle \mathcal{H}_N^{MF} \rangle$ is subtracted in which we use the normalization with $\langle \mathcal{H}_N^{MF}(T=0, \rho_N=0) \rangle = 0$. The values of the model parameters for nuclear matter are summarized in Table I. As may be seen in this table, the momentum cutoff $\Lambda_N$ is rather small. It is possible, however, that the cutoff similarly increases with density by considering a chemical potential-dependent cutoff $\Lambda(\mu)$ [45] [46]. In Fig. 2-1, the saturation curve is shown under the pa-
Table I: The parameter set for nuclear matter \((i = N)\).

<table>
<thead>
<tr>
<th>(\Lambda_N [\text{MeV}])</th>
<th>(G_N^N \Lambda_N^2)</th>
<th>(G_N^N \Lambda_N^2)</th>
<th>(G_N^N \Lambda_N^8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>377.8</td>
<td>19.2596</td>
<td>17.9824</td>
<td>-1069.89</td>
</tr>
</tbody>
</table>

Fig. 2-1: Nuclear matter saturation properties with \(m_N(\rho_N^0) = 0.6m_N(\rho_N = 0)\), in which the incompressibility is evaluated as \(K \approx 260 \text{ MeV}\).

The parameters of Table I. Numerically, a rather reasonable incompressibility is obtained, around 260 MeV, in which the incompressibility of nuclear matter at normal nuclear density is evaluated as

\[ K = 9\rho_N^0 \cdot d^2W_N(\rho_N)/d\rho_N^2|_{\rho_N = \rho_N^0} \].

In this thesis, although we fix the value of \(m_N(\rho_N^0)\), the incompressibility \(K\) at normal nuclear density is also capable of becoming an input parameter instead of the nucleon mass \(m_N(\rho_N^0)\) at normal nuclear density since \(m_N(\rho_N^0)\) has an influence on \(K\).

As for quark matter, in the extended NJL model with scalar-vector eight-point interaction, there are three parameters: \(G_s^q, G_v^q\) and \(\Lambda_q\). Here, we put \(G_v^q = 0\) since the effects of the vector coupling \(G_v^q\) are well-known. On the other hand, we introduce the parameter \(G_s^q\) since it has an influence on the chiral phase transition at finite density and temperature. The scalar-vector attractive interaction with \(G_s^q\) increases the chiral condensate strength. Namely, the chiral phase transition point is pushed to the higher-density side with increasing \(G_s^q\), in which the scalar coupling \(G_s^q\) is effectively regarded as a density-dependent coupling \(G_s^q(\rho_q)\) by introducing \(G_s^q\). In this paper, we treat \(G_s^q\) as a chiral phase transition tuning parameter. The parameters \(G_s^q\) and \(\Lambda_q\) are determined by two conditions: the vacuum value for the dynamical quark mass and the pion decay constant are taken so as to reproduce \(m_q = 313 \text{ MeV}\) and \(f_\pi = 93 \text{ MeV}\), respectively. For the value of \(G_s^q\), there is no criterion to determine it in this stage. Thus, we treat \(G_s^q\) as a free parameter. The physical quantities of quark matter case are obtained from the corresponding ones for the nuclear matter case with \(i = N \rightarrow i = q\). Namely, \(m_N \rightarrow m_q\), \(\rho_N \rightarrow \rho_q\) and \(\nu_N = 2N_f N_c^N \rightarrow \nu_q = 2N_f N_c^q\) with \(N_f = 2, N_c^q = 3\). The values of model-parameters for quark

\[ m_q = -2G_s^q (\bar{\psi}_q \psi_q) = \frac{2}{\pi^2} G_s^q N_f N_c m_q f_\pi^4 \int_0^{\Lambda_q^4} d(p^2) \frac{p^2}{\sqrt{p^2 + m_q^2}} \].

\[ f_\pi^2 = \frac{1}{2\pi^2} N_c m_q^2 \int_0^{\Lambda_q^4} d(p^2) \frac{p^2}{(p^2 + m_q^2)^{3/2}} \].
Table II: The parameter set for quark matter ($i = q$).

<table>
<thead>
<tr>
<th>$\Lambda_q$[MeV]</th>
<th>$G_q^2 \Lambda_q^2$</th>
<th>$G_q^2 \Lambda_q^2$</th>
<th>$G_q^2 \Lambda_q^8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>653.961</td>
<td>2.130 22</td>
<td>0</td>
<td>free</td>
</tr>
</tbody>
</table>

matter are summarized in Table II.
3.1 Gap solutions

In the quark matter case, the coupling constant for the scalar-vector and isoscalar-vector eight-point interaction, $G^q_{sv}$, is free to choose the value, as already mentioned in \( \text{[2.1.2]} \) Thus, by varying the strength of the scalar-vector coupling, we investigate the $G^q_{sv}$-dependence on the phase transition or phase diagram. Since $G^q_{sv}$ is a free parameter, we take some patterns of $G^q_{sv}$-values. In the $G^q_{sv} = 0$ case, the dynamical quark mass at normal nuclear density, $m_q(\rho_N)$, can be numerically calculated as 187 MeV by using the original NJL model with $G^q_{sv} = G^q_{sv} = 0$, namely, $m_q(\rho_N^0 = 0.17/\text{fm}^3) \approx 187$ MeV which is approximately evaluated as 0.6$m_q(\rho_N = 0)$ where $m_q(\rho_N = 0) = 313$ MeV. Hence, we tune the parameter $G^q_{sv}$ by varying the value 0.6 as the ratio of the in-medium to the vacuum quark mass. Here, we first choose the parameter $G^q_{sv}$ as $m_q(\rho_N^0) = 0.625m_q(\rho_N = 0)$, which leads to $G^q_{sv}\Lambda_q^8 = -68.4$. In this section, we present the results with $G^q_{sv}\Lambda_q^8 = -68.4$ as a typical case in which the chiral phase transition is realized at a reasonable point \( \text{[42]} \), while we take three patterns of $G^q_{sv}$-values: $G^q_{sv} = 0$, $G^q_{sv}\Lambda_q^8 = -68.4$ and $G^q_{sv}\Lambda_q^8 = -81.9$. The results with other $G^q_{sv}$-values are summarized in Appendix A.

Below, we present the numerical results with $G^q_{sv}\Lambda_q^8 = -68.4$. The left-hand side of Fig. 3-1 shows the dynamical quark mass as a function of the quark chemical potential at $T = 0, 20$ and 40 MeV. For zero temperature, the obtained vacuum value for the dynamical quark mass is $\mu_q = 313$ MeV. As may be seen in this figure, the region with multiple solutions for the gap equation shrinks with increasing temperature.

The right-hand side of Fig. 3-1 shows the quark number density as a function of the quark chemical potential at $T = 0, 20$ and 40 MeV. Here, the quark number density is given as a multiple of normal nuclear density $\rho_N^0$ in the vertical axis. The relation between the quark number density $\rho_q$ and the quark chemical potential $\mu_q$ is derived from Eq. \( \text{[2.8]} \). The solid curves and branch lines show stable solutions in this figure. Here, the solid branch lines represent the massless solutions of quark number density, $\rho_q(m_q=0)$. On the other hand, the dashed curves denote the unphysical region.
**Fig. 3-1**: Left: The dynamical quark mass $m_q$ as a function of the quark chemical potential $\mu_q$ with $G_{\text{sym}}^q \Lambda_q^8 = -68.4$ at $T = 0, 20$ and 40 MeV. Right: The quark number density divided by normal nuclear density, $p_q/\rho_N^0$, as a function of the quark chemical potential $\mu_q$ with $G_{\text{sym}}^q \Lambda_q^8 = -68.4$ at $T = 0, 20$ and 40 MeV. The solid curves and branch lines are stable solutions, while the dashed curves are unstable solutions.

### 3.2 Chiral phase transition in pressure comparison

The gap equation has multiple solutions in certain regions, and hence an unphysical region with unstable solutions appears in the quark phase. Thus, in order to determine the physically realized solution, we calculate the pressure in Eq. (2.16) and compare the pressures with gap solutions and with massless solutions at the same chemical potential, and among them, the largest pressure corresponds to the physically realized solution, namely, the solution with the largest pressure is physically realized at each temperature. Figure 3-2 shows the pressure of quark matter, $p_q$, as a function of the quark chemical potential $\mu_q$.

On the right-hand side of Fig. 3-1 at $T = 0$ MeV, the lower density solution is realized from $\mu_q = 313$ MeV to $\mu_q \approx 326$ MeV, where the pressure takes the largest value in Fig. 3-2. Above $\mu_q \approx 326$ MeV, however, the massless solution is physically realized. Namely, the chiral phase transition for $T = 0$ MeV occurs at $\mu_q \approx 326$ MeV in Fig. 3-2. Here, the phase with dynamical quark mass, which is the chiral broken phase, is realized for the lower quark chemical potential, $\mu_q < 326$ MeV, and the phase with massless quark, which is the chiral symmetric phase, for the higher one, $\mu_q > 326$ MeV as may be seen in Fig. 3-2. In the region from $\rho_q \sim 0.38\rho_N^0 \ (\rho_B \sim 0.13\rho_N^0)$ where $\rho_N(= \rho_N = \rho_q/3)$ represents the baryon number density, to $\rho_q \sim 5.41\rho_N^0 \ (\rho_B \sim 1.80\rho_N^0)$ on the right-hand side of Fig. 3-1, a first-order chiral phase transition is realized and the coexistence of quark phases occurs.

In the case of $T = 20$ MeV, the low density solution is realized up to $\mu_q \approx 323$ MeV and the massless solution becomes physically realized from $\mu_q \approx 323$ MeV on the right-hand side of Fig. 3-1. From Fig 3-2, it is seen that the chiral phase transition with $T = 20$ MeV occurs at $\mu_q \approx 323$ MeV. In this case, the chiral broken phase is realized at $\mu_q < 323$ MeV and the chiral symmetric phase at $\mu_q > 323$ MeV. Here, in the region from $\rho_q \sim 1.30\rho_N^0 \ (\rho_B \sim 0.43\rho_N^0)$ to $\rho_q \sim 5.41\rho_N^0 \ (\rho_B \sim 1.80\rho_N^0)$
Fig. 3-2: The pressure of quark matter $p_q$ as a function of the quark chemical potential $\mu_q$ with $G_{\pi^0}^q A_q^8 = -68.4$ at $T = 0, 20$ and $40$ MeV. The solid lines represent the pressure with gap solution, while the dash-dotted lines represent the pressure with massless solution. The chiral phase transitions at $T = 0, 20$ and $40$ MeV occur at $\mu_q \approx 326, 323$ and $318$ MeV, respectively.

on the right-hand side of Fig. 3-1, a first-order chiral phase transition is realized and coexistence of quark phases occurs.

In the case of $T = 40$ MeV, the density solution with gap solution is realized in all points (from $\mu_q = 0$ to $\mu_q \approx 318$ MeV) and the massless solution becomes physically realized from $\mu_q \approx 318$ MeV on the right-hand side of Fig. 3-1. In Fig. 3-2 with $T = 40$ MeV, two branches are smoothly connected and at $\mu_q \approx 318$ MeV a chiral phase transition occurs, or at $\rho_q \sim 5.78 \rho_N^0 \rho_B \sim 1.93 \rho_N^0$ on the right-hand side of Fig. 3-1. In this case, the phase with chiral symmetry breaking is realized for $\mu_q < 318$ MeV and the phase with chiral restoration for $\mu_q > 318$ MeV. Unlike in the case of $T = 0$ or $20$ MeV, at $T = 40$ MeV, there is no jump from the lower density solution to the massless one since there is no unstable density solution as may be seen on the right-hand side of Fig. 3-1 with $T = 40$ MeV. Hence, coexistence of quark phases does not occur and the order of the chiral phase transition is not a first-order phase transition. In the case in which a jump exists, as in the right-hand side of Fig. 3-1 with $T = 0$ or $20$ MeV, it is clear that the phase transition is of first order. However, since there is no jump on the right-hand side of Fig. 3-1 with $T = 40$ MeV and the slope of the pressure with respect to quark chemical potential is the same, namely, the two branches are smoothly connected as seen in Fig. 3-2, the phase transition at $T = 40$ MeV is of second order.

Incidentally, the chiral symmetry restoration occurs systematically at a rather low density in comparison with the quark-hadron phase transition as will be discussed later, in which the critical densities of the chiral and quark-hadron phase transitions are different. Thus, it has no actual physical consequences and no influence on the curve of the equation of state in the quark phase after the quark-hadron phase transition.\(^1\)

\(^1\) With increasing the coupling $G_{\pi^0}^q$, the two critical points are approach each other since the critical density of the chiral phase transition is pushed to higher density side by the strengthening of the scalar-vector interaction, while
3.3 Quark-hadron phase transition in pressure comparison

To investigate the quark-hadron phase transition is the main object of this thesis in the extended NJL model with scalar-vector eight-point interaction at finite temperature and density. In this section, we present a procedure for describing the quark-hadron phase transition between the nuclear and quark matters and show the numerical results. For the quark-hadron phase transition, we use the pressure comparison approach, as has already been shown in the previous section, namely, we determine the realized phase by comparing the pressure of nuclear matter with that of quark matter at finite temperature and baryon chemical potential. For this purpose, the condition for chemical equilibrium is required to be

\[ \mu_N(T) = 3\mu_q(T), \]

where \( \mu_N \) and \( 3\mu_q \) indicate the chemical potential per baryon. By regarding this condition as the one for thermodynamic equilibrium between the hadron and quark phases, we derive the corresponding condition for the pressure of hadron and quark phases as

\[ p_N(\mu_N, T) = p_q(3\mu_q, T). \]

From Eq. (3.1), the pressure \( p_N \) and \( p_q \) can be calculated for nuclear matter and for quark matter, respectively.\(^\text{2}\)

The left-hand side of Fig. 3-3 shows the pressures of nuclear matter and quark matter as a function of the baryon chemical potential, which is equivalent to the nuclear chemical potential and the triple of the quark chemical potential at \( T = 0 \) in the case of \( G_s v \Lambda_q^6 = -68.4 \). As may be seen from this figure, there is a crossing point at a certain chemical potential value. From this crossing point, we can determine the coexistence of nuclear and quark phases. Then, about \( \mu_B \approx 1236 \) MeV, the quark-hadron phase transition at \( T = 0 \) occurs on the left-hand side of Fig. 3-3. Thus, the nuclear phase or hadron phase is realized for a smaller baryon chemical potential, \( \mu_B(= \mu_N) < 1236 \) MeV, and the quark phase for a larger chemical potential, \( \mu_B(= 3\mu_q) > 1236 \) MeV.

On the other hand, the right-hand side of Fig. 3-3 shows the pressure as functions with respect to the baryon number density at \( T = 0 \) where the vertical axis is shown on a logarithmic scale. This figure is depicted by using the relation of the density and chemical potential shown in the left-hand side of Fig. 3-1. It is seen from this figure that the nuclear and quark phases coexist and a first-order quark-hadron phase transition occurs in the region from \( \rho_N(= \rho_B) \sim 2.64\rho_N^0 \) to \( \rho_q \sim 10.9\rho_N^0 \) (\( \rho_B \sim 3.63\rho_N^0 \)).

In the case of \( T = 20 \) MeV, it is seen that the quark-hadron phase transition occurs at \( \mu_B \approx 1190 \) MeV, as may be seen from the left-hand side of Fig. 3-4. Thus, the nuclear or hadron phase is realized in the region of \( \mu_N < 1190 \) MeV and the quark phase in the region of \( 3\mu_q > 1190 \) MeV. From the right-hand side of Fig. 3-4, it is seen that coexistence of nuclear and quark phases occurs

\(^*\) the critical density of the quark-hadron phase transition is not changed.

\(^\text{2}\) We only require the baryon number conservation in which for a single conserved charge the transition occurs at constant pressure.
Fig. 3-3: The quark-hadron phase transition is shown at $T = 0$ in the case of $G_A^q \Lambda_N^q = -68.4$. Left: The pressure of nuclear matter ($i = N$) and of quark matter ($i = q$) as functions with respect to the baryon chemical potential $\mu_B$ ($= \mu_N = 3\mu_q$) at $T = 0$ MeV. Right: The pressure $p_i/p_0$ as a function of the baryon number density $\rho_B/\rho_N^0$ at $T = 0$ MeV. Here, the vertical axis is shown on a logarithmic scale. The pressure is divided by $p_0$, where $p_0 = 1.0$ MeV/fm$^3$. The baryon number density is given in multiples of normal nuclear density $\rho_N^0$.

Fig. 3-4: The quark-hadron phase transition is shown at $T = 20$ MeV with $G_A^q \Lambda_N^q = -68.4$. Left: The pressures of nuclear matter and quark matter as functions with respect to the baryon chemical potential $\mu_B$ at $T = 20$ MeV. Right: The pressure is presented on a logarithmic scale $p_i/p_0$ as a function of the baryon number density $\rho_B/\rho_N^0$ at $T = 20$ MeV.
Fig. 3-5: The first-order quark-hadron phase transition disappears at \( T = 40 \text{ MeV} \) in the extended NJL model with \( G^q_{4v}A_q^4 = -68.4 \). Left: The pressure \( p_i \) as a function of the baryon chemical potential \( \mu_B \) at \( T = 40 \text{ MeV} \). Right: The pressure presented on a logarithmic scale \( p_i/p_0 \) as a function of the baryon number density \( \rho_B/\rho_N^0 \) at \( T = 40 \text{ MeV} \).

from \( \rho_N (=\rho_B) \sim 2.49\rho_N^0 \) to \( \rho_q \sim 9.99\rho_N^0 \) (\( \rho_B \sim 3.33\rho_N^0 \)) and a first-order chiral phase transition is realized there. As compared with the case of \( T = 0 \), it is seen from Fig. 3-4 that the crossing point, namely, the quark-hadron phase transition point, moves toward the lower left side, concretely, small chemical potential and low pressure side, and the density region of coexistence of nuclear and quark phases becomes smaller with increasing temperature.

In the process of increasing temperature, at a certain temperature, the crossing point vanishes. This situation is depicted in Fig. 3-5 with \( T = 40 \text{ MeV} \). In this figure, there is no crossing point that represents the transition point, so this leads to the situation in which the first-order quark-hadron phase transition has already finished.
Chapter 4

Quark-Hadron Phase Boundary

4.1 $G^q_{sv}$-dependence on the phase diagram

In this chapter, we present a phase diagram in the extended NJL model with scalar-vector eight-point interaction at finite temperature and baryon chemical potential. Moreover, in the quark matter, we investigate the effects of the scalar-vector coupling constant $G^q_{sv}$, which has an influence on the chiral phase transition in the phase diagram. Namely, we discuss how the phase boundary of the chiral phase transition is affected by varying the strength of the scalar-vector interaction.

4.1.1 Phase diagram with $G^q_{sv} = 0$

First, we present the phase diagram without scalar-vector interaction. Thus, for the quark matter, we adopt the original NJL model Lagrangian density without the scalar-vector interaction, that is, $G^q_{sv} = 0$ and describe the chiral and quark-hadron phase transitions.

The top panel of Fig. 4-1 shows the phase diagram with $G^q_{sv} = 0$ as a function of temperature and baryon chemical potential. Here, the vertical and horizontal axes represent temperature $T$ and baryon chemical potential $\mu_B$ ($= \mu_N = 3\mu_q$), respectively. The solid curve represents the critical line of the first-order chiral phase transition and the dotted one represents that of the second-order chiral phase transition. As may be seen from this figure, the critical line of the first-order chiral phase transition emerging from a point in the $T = 0$ and $\mu_B \approx 978$ MeV terminates at $(\mu_B, T) \simeq (842, 80)$ MeV. Also, the critical temperature at vanishing chemical potential, $\mu_B = 0$, is found to be about 190 MeV. The dash-dotted curve represents the first-order quark-hadron phase transition. In this figure, it is seen that an endpoint of the first-order quark-hadron transition occurs at $(\mu_B, T) \simeq (955, 39)$ MeV.

4.1.2 Phase diagram with $G^q_{sv} \Lambda_q^8 = -68.4$

Next, we show the phase diagram with scalar-vector interaction. For the scalar-vector interaction strength, we consider the value $G^q_{sv} \Lambda_q^8 = -68.4$ used in the previous chapter.
Fig. 4-1: The phase diagrams in the extended NJL model with $G_{sv}^3 = 0$ (top), $G_{sv}^3 \Lambda_s^8 = -68.4$ (middle) and $G_{sv}^3 \Lambda_s^8 = -81.9$ (bottom) are depicted as a temperature-baryon chemical potential ($T$-$\mu_B$) plane. The solid, dotted and dash-dotted curves indicate the first-order chiral phase transition, the second-order chiral phase transition and the first-order quark-hadron phase transition, respectively. The endpoint of the 1st-order chiral phase transition is at $(\mu_B, T) \approx (842, 80)$ MeV (top), $(\mu_B, T) \approx (964, 26)$ MeV (middle) and $(\mu_B, T) \approx (979, 1)$ MeV (bottom). For the first-order quark-hadron phase transition, the position of the endpoint is at $(\mu_B, T) \approx (955, 39)$ MeV.
The middle panel of Figure 4-1 shows the phase diagram in the $T$-$\mu_B$ plane with $G_{sv}^q A_q^8 = -68.4$. The critical line of the first-order chiral phase transition is depicted as a solid curve and that of the first-order quark-hadron phase transition as a dash-dotted curve. The dotted curve represents the critical line of the second-order chiral phase transition. The critical line of the first-order chiral phase transition emerges at $T = 0$ and $\mu_B \approx 979$ MeV and terminates at $(\mu_B, T) \simeq (964, 26)$ MeV. For the critical line of the first-order quark-hadron phase transition emerging from a point in $T = 0$ and $\mu_B \approx 1236$ MeV, it terminates at $(\mu_B, T) \simeq (955, 39)$ MeV. Then, the endpoint of the first-order quark-hadron phase transition is located on the critical line of the chiral phase transition, as may be seen from the middle panel of Fig. 4-1.

Here, it should be noted that there is a region where the quark-hadron phase transition occurs after chiral symmetry restoration in the nuclear phase side, in which the nucleon mass is zero. Namely, this suggests that a phase that is chiral symmetric but with nucleonic/hadronic elementary excitation would exist just before the phase transition from the nuclear phase to the quark one. Recently, McLerran and Pisarski have proposed a new state of matter, the so-called quarkyonic matter $^{[2]}$, which is a phase characterized by chiral symmetry restoration and confinement based on large $N_c$ arguments. Thus, this chiral symmetric nuclear phase predicted by our model may possibly correspond to the quarkyonic phase.

### 4.1.3 Phase diagram with $G_{sv}^q A_q^8 = -81.9$

Finally, we show the phase diagram with the stronger scalar-vector interaction. We set the value $G_{sv}^q A_q^8 = -81.9$, in which $m_q(\rho_q/3 = \rho_N^0) = 0.63m_q(\rho_q = 0)$.

The bottom panel of Figure 4-1 shows the phase diagram in the $T$-$\mu_B$ plane with $G_{sv}^q A_q^8 = -81.9$. The critical line of the first-order chiral phase transition, depicted as a solid curve, emerges at $T = 0$ and $\mu_B \approx 979$ MeV and soon terminates at $(\mu_B, T) \simeq (979, 1)$ MeV. As for the critical line of the first-order quark-hadron phase transition depicted as a dash-dotted curve, the same diagram as the previous two is obtained.

The reason why the behavior of the first-order quark-hadron phase transition line seen in Fig. 4-1, does not change is that the critical line of the first-order quark-hadron phase transition exists in the chiral symmetric phase, in which $m_q = 0$ and $\langle \overline{\psi}_q \psi_q \rangle = 0$, that is, the quark-hadron phase transition occurs after the chiral phase transition for the unstable quark phase with $G_{sv}^q = 0$, $G_{sv}^q A_q^8 = -68.4$ and $G_{sv}^q A_q^8 = -81.9$. Here, in the expression for pressure $p_q$ in Eq. (2.16) there is no $G_{sv}^q$-dependence since $\langle \mathcal{H}_{\text{MF}}^q \rangle$ in Eq. (2.12) and $\mu_q^x$ in Eq. (2.6) do not depend on $G_{sv}^q$, due to $\langle \overline{\psi}_q \psi_q \rangle = 0$.

From Fig. 4-1, by varying the strength of $G_{sv}^q$, it is seen that the critical line of first-order chiral phase transition shrinks with increasing $G_{sv}^q$. Namely, $G_{sv}^q$ acts to move the endpoint of the first-order chiral phase transition toward larger baryon chemical potentials and lower temperatures. In the case of $G_{sv}^q A_q^8 = -68.4$, the endpoint of the first-order quark-hadron phase transition is located on the chiral phase transition line. Also, the chiral symmetry restoration is shifted toward a larger baryon chemical potential. In the case of the stronger $G_{sv}^q$, the line of the first-order chiral
phase transition disappears.
Chapter 5

Concluding Remarks

5.1 Summary and future prospects

The quark-hadron phase transition at finite temperature and baryon chemical potential has been investigated following Ref [42] in the extended NJL model with scalar-vector eight-point interaction. In this model, as a first attempt to investigate the quark-hadron phase transition, the hadron side was regarded as symmetric nuclear matter and the quark side as a free quark phase with no quark-pair correlation. Here, the single nucleon in the nuclear matter and the single quark in the quark matter were treated as fundamental fermions with \( N_c^N = 1 \) and \( N^q_0 = 3 \), respectively. Then, in this model, the nuclear saturation property has been well reproduced for the nuclear matter side. On the other hand, for the quark matter side, there is one free parameter, \( G^q_\text{sv} \). This model parameter was not fixed in this paper, since there is no criterion to determine the value of \( G^q_\text{sv} \) by using physical quantities at this stage. By introducing this parameter \( G^q_\text{sv} \), the effective density-dependent coupling constant was obtained as \( G^q_\text{sv}(\rho_q) = G^q_\text{sv}(1 - G^q_\text{sv}/G^q_\text{sv} \cdot \rho^q_q) \). Hence, the \( G^q_\text{sv} \) term plays an important role in pushing the chiral symmetry restoration point to higher density side for quark matter. Thus, the parameter \( G^q_\text{sv} \) controls the chiral symmetry restoration point and/or the strength of the partial chiral symmetry restoration in the nuclear medium. As for the description of the quark-hadron phase transition, we have calculated the pressure of the nuclear matter and the quark matter, and determined the realized phase by comparing their pressures. As a result, a first-order quark-hadron phase transition is obtained at finite temperature and baryon chemical potential. Here, the end-point of the first-order quark-hadron phase transition is at \((\mu_B, T) \approx (955, 39)\) MeV with \( G^q_\text{sv} A^8_q = -68.4 \). This phase boundary is not changed even by varying the strength of the scalar-vector interaction because of \( G^q_\text{sv} \)-independence. As for the effects of the scalar-vector coupling constant \( G^q_\text{sv} \) on the chiral phase transition, the critical line of the first-order chiral phase transition shrinks with increasing \( G^q_\text{sv} \). Namely, \( G^q_\text{sv} \) acts to move the endpoint of the first-order chiral phase transition toward larger \( \mu_B \) and lower \( T \).

From the phase diagram in Fig. 4-1, it should be noted that there is an interesting phase where the quark-hadron phase transition occurs after chiral symmetry restoration in the nuclear matter. Figure 5-1 shows an enlargement of quarkyonic regime in Fig. 4-1. This might appear as an
Fig. 5-1: The phase diagram with a quarkyonic regime where a state with chiral symmetry restoration but an elementary excitation is nucleonic, which is an enlargement of intermediate-density region in Fig. 4-1.

exotic phase, that is, the nuclear phase, not the quark phase, while the chiral symmetry is restored in terms of the quark matter. This phase may possibly correspond to the quarkyonic phase [52], which is introduced as a chiral symmetric confined matter. Also, in Fig. 5-1, there may appear a region that looks like an approximate triple point where a nuclear phase, a free quark phase, and quarkyonic-like phase meet. In Ref. [52], it is speculated that a hadronic phase, a quarkyonic phase, and a quark-gluon plasma phase meet at a triple point in the QCD phase diagram. Here, this triple point is located at a limiting temperature of the chemical freeze-out, in which strange particle multiplicity ratios, i.e., strange to non-strange particle ratios, e.g., $K^+/\pi^+$, exhibit non-monotonic behavior. When assuming the existence of such triple point and of quarkyonic phase in the QCD phase diagram, it is suggested that the observed statistical properties of experimental data in ultra-relativistic nuclear collisions are naturally explained. On the other hand, if we consider the Polyakov-loop dynamics, it would have been possible to present a more convincing argument about the quarkyonic phase. In Appendix C, the Polyakov-loop potential, not covered here, is introduced as one of additional terms in the quark NJL model.

In this paper, we have ignored the color superconducting phase [53]. However, this phase may exist in finite density systems. Thus, the next challenging task might be to investigate the phases of nuclear matter, including nuclear superfluidity and quark matter, and also including the color superconducting state, i.e., nucleon pairing on the nuclear phase side and quark pairing on the quark phase side. As for the two-flavor superconducting (2SC) state, the quark pairing is introduced in Appendix C. Further, it is widely believed that neutron star matter undergoes a phase transition to quark matter at high densities. Therefore, it is also interesting to investigate the phase transition between neutron matter or $\beta$-equilibrium matter and quark matter. This leads to the understanding and development of the physics of neutron stars. In Appendix D we refer to it.
APPENDIX A

Numerical Results for Different $G_{SV}^{q}$

A.1 The case of $G_{SV}^{q} = 0$

In the original NJL model with $G_{SV}^{q} = 0$, the dynamical quark mass at normal nuclear density can be numerically calculated as

$$m_q(\rho_N^0 = 0.17/[\text{fm}^3]) \approx 187 \text{ MeV}.$$  \hspace{1cm} (A-1)

Thus, it is evaluated that $m_q(\rho_N^0)/m_q(\rho_N = 0) \approx 0.6$ where $m_q(\rho_N = 0) = 313 \text{ MeV}$.

By solving the gap equation in Eq. (2.3) with $G_{SV}^{q} = 0$, the dynamical quark mass $m_q$ is obtained at finite temperature and quark chemical potential. The upper left-hand side panel of Fig. A-1 shows the dynamical quark mass as a function of the quark chemical potential at $T = 0, 50$ and 100 MeV, in which $m_q(T = 0, \mu_q = 313) = 313 \text{ MeV}$. Here, the region with multiple solutions for the gap equation shrinks with increasing temperature in this figure.

On the other hand, the upper right-hand side panel of Fig. A-1 shows the quark number density as a function of the quark chemical potential at $T = 0, 50$ and 100 MeV, in which the quark number density is given as a multiple of normal nuclear density $\rho_N^0$ in the vertical axis. Here, we obtain the relation between the quark number density $\rho_q$ and the quark chemical potential $\mu_q$ from Eq. (2.8). In this figure, the solid curves and branch lines show stable solutions, where the solid branch lines represent the massless solutions of quark number density $\rho_q(\mu_q=0)$, while the dashed curves denote the unphysical region.

Since the gap equation has multiple solutions in certain regions and hence an unphysical region with unstable solutions appears in the quark phase, we calculate the pressure in Eq. (2.16) and compare the pressures with gap solutions and with massless solutions at the same chemical potential in order to determine the physically realized solution which leads to the largest pressure. The bottom panel of Fig.A-1 shows the pressure of quark matter $p_q$, as a function of the quark chemical potential $\mu_q$.

\footnote{Incidentally, in the nuclear matter case, we impose the condition, $m_N(\rho_N^0) = 0.6m_N(\rho_N = 0) \approx 563 \text{ MeV}$ with $m_N(\rho_N^0) = 939 \text{ MeV}$, which leads to $m_N(\rho_N^0) \approx 3m_q(\rho_q^0)$ \cite{13}.}

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Fig. A-1: Upper left-hand side panel: The dynamical quark mass $m_q$ as a function of the quark chemical potential $\mu_q$ with $G^0_{su} = 0$ at $T = 0, 50,$ and 100 MeV. Upper right-hand side panel: The quark number density divided by normal nuclear density, $\rho_q/\rho^0_N$, as a function of the quark chemical potential $\mu_q$ with $G^0_{su} = 0$ at $T = 0, 50,$ and 100 MeV. The solid curves and branch lines are stable solutions, while the dashed curves are unstable solutions. Bottom panel: The pressure of quark matter $p_q$ as a function of the quark chemical potential $\mu_q$ with $G^0_{su} = 0$ at $T = 0, 50,$ and 100 MeV. The solid lines represent the pressure with gap solution, while the dash-dotted lines represent the pressure with massless solution. The chiral phase transitions at $T = 0, 50,$ and 100 MeV occur at $\mu_q \approx 326, 305$ and 263 MeV, respectively.
On the upper right-hand side panel of Fig. A-1 at $T = 0$ MeV, the lower density solution is realized from $\mu_q = 313$ MeV to $\mu_q \approx 326$ MeV. Above $\mu_q \approx 326$ MeV, the massless solution is physically realized, where the chiral phase transition for $T = 0$ MeV occurs at $\mu_q \approx 326$ MeV. As may be seen in this panel, the phase with dynamical quark mass, which is the chiral broken phase, is realized for the lower quark chemical potential, $\mu_q < 326$ MeV, and the phase with massless quark, which is the chiral symmetric phase, for the higher one, $\mu_q > 326$ MeV. In the region from $\rho_q \sim 0.28\rho_N^0 (\rho_B \sim 0.09\rho_N^0)$ to $\rho_q \sim 5.41\rho_N^0 (\rho_B \sim 1.80\rho_N^0)$ on the upper right-hand side panel of Fig. A-1 at $T = 0$ MeV, a first-order chiral phase transition is realized and coexistence of quark phases occurs.

In the case of $T = 50$ MeV, the low density solution is realized up to $\mu_q \approx 305$ MeV and the massless solution becomes physically realized from $\mu_q \approx 305$ MeV on the bottom of Fig A-1. From this figure, it is seen that the chiral phase transition with $T = 50$ MeV occurs at $\mu_q \approx 305$ MeV. In this case, the chiral broken phase is realized at $\mu_q < 305$ MeV and the chiral symmetric phase at $\mu_q > 305$ MeV. Here, in the region from $\rho_q \sim 2.76\rho_N^0 (\rho_B \sim 0.92\rho_N^0)$ to $\rho_q \sim 5.57\rho_N^0 (\rho_B \sim 1.86\rho_N^0)$ on the upper right-hand side panel of Fig. A-1, a first-order chiral phase transition is realized and the coexistence of quark phases occurs.

In the case of $T = 100$ MeV, the density solution with gap solution is realized in all points (from $\mu_q = 0$ to $\mu_q \approx 263$ MeV) and the massless solution becomes physically realized from $\mu_q \approx 263$ MeV on the upper right-hand side of Fig. A-1. On the bottom panel of Fig. A-1 with $T = 100$ MeV, two branches are smoothly connected and at $\mu_q \approx 263$ MeV a chiral phase transition occurs, or at $\rho_q \sim 6.33\rho_N^0 (\rho_B \sim 2.11\rho_N^0)$ on the upper right-hand side panel of Fig. A-1. In this case, the phase with chiral symmetry breaking is realized for $\mu_q < 263$ MeV and the phase with chiral restoration for $\mu_q > 263$ MeV. As may be seen on the upper right-hand side of Fig. A-1 with $T = 100$ MeV, there is no jump from the lower density solution to the massless one since there is no unstable density solution. Therefore, the phase transition at $T = 100$ MeV is of second order.

### A.2 The case of $G_{sv}^q \Lambda_q^8 = -81.9$

We put $m_N(\rho_N^0) = 0.63 m_N(\rho_N = 0)$, which leads to $G_{sv}^q \Lambda_q^8 = -81.9$. To obtain the dynamical quark mass $m_q$, we solve the gap equation in Eq. (23) with $G_{sv}^q \Lambda_q^8 = -81.9$ at finite temperature and quark chemical potential.

The upper left-hand side panel of Fig. A-2 shows the dynamical quark mass as a function of the quark chemical potential at $T = 0$ and 30 MeV, where the region with multiple solutions for the gap equation shrinks with increasing temperature, while the upper right-hand side panel of Fig. A-1 shows the quark number density as a function of the quark chemical potential at $T = 0$ and 30 MeV, in which the quark number density is given as a multiple of normal nuclear density $\rho_N^0$ in the vertical axis. Here, the solid curves and branch lines show stable solutions, and the dashed curves denote the unphysical region.

To determine the physically realized solution which leads to the largest pressure, we compare
Fig. A-2: Upper left-hand side panel: The dynamical quark mass $m_q$ as a function of the quark chemical potential $\mu_q$ with $G_{\pi}^qA_q^8 = -81.9$ at $T = 0$ and 30 MeV. Upper right-hand side panel: The quark number density divided by normal nuclear density, $\rho_q/\rho_N^0$, as a function of the quark chemical potential $\mu_q$ with $G_{\pi}^qA_q^8 = -81.9$ at $T = 0$ and 30 MeV. The solid curves and branch lines are stable solutions, while the dashed curves are unstable solutions. Bottom panel: The pressure of quark matter $p_q$ as a function of the quark chemical potential $\mu_q$ with $G_{\pi}^qA_q^8 = -81.9$ at $T = 0$ and 30 MeV. The solid lines represent the pressure with gap solution, while the dash-dotted lines represent the pressure with massless solution. The chiral phase transitions at $T = 0$ and 30 MeV occur at $\mu_q \approx 326$ and 323 MeV, respectively.
the pressures with gap solutions and with massless solutions at the same chemical potential. The bottom panel of Fig. A-2 shows the pressure of quark matter $p_q$, as a function of the quark chemical potential $\mu_q$.

In the case of $T = 0$ MeV, the low density solution is realized up to $\mu_q \approx 326$ MeV and the massless solution becomes physically realized from $\mu_q \approx 326$ MeV on the bottom of Fig. A-2. From this figure, it is seen that the chiral phase transition with $T = 0$ MeV occurs at $\mu_q \approx 326$ MeV. In this case, the chiral broken phase is realized at $\mu_q < 326$ MeV and the chiral symmetric phase at $\mu_q > 326$ MeV. Here, a first-order chiral phase transition is realized and coexistence of quark phases occurs in the region from $\rho_q \sim 0.29\rho_N^0$ ($\rho_B \sim 0.10\rho_N^0$) to $\rho_q \sim 5.43\rho_N^0$ ($\rho_B \sim 1.81\rho_N^0$) on the upper right-hand side panel of Fig. A-2.

In the case of $T = 30$ MeV, the density solution with gap solution is realized in all points (from $\mu_q = 0$ to $\mu_q \approx 323$ MeV) and the massless solution becomes physically realized from $\mu_q \approx 323$ MeV on the upper right-hand side Fig. A-2. On the bottom panel of Fig. A-2 with $T = 30$ MeV, two branches are smoothly connected and at $\mu_q \approx 323$ MeV a chiral phase transition occurs, or at $\rho_q \sim 5.71\rho_N^0$ ($\rho_B \sim 1.90\rho_N^0$) on the upper right-hand side panel of Fig. A-2. In this case, the phase with chiral symmetry breaking is realized for $\mu_q < 323$ MeV and the phase with chiral restoration for $\mu_q > 323$ MeV. Here, there is no jump from the lower density solution to the massless one, as may be seen on the upper right-hand side of Fig. A-2 with $T = 30$ MeV. Thus, the phase transition at $T = 30$ MeV is of second order.
Appendix B

Numerical Treatment

B.1 Numerical recipes

In this appendix, we fill in with details on the numerical calculations. As already written in Chapter 2, the NJL gap equation (2.3) with Eqs. (2.6)-(2.10) are

$$ m_i = -2G_s^i \left[ 1 - \frac{G_{sv}^i}{G_s^i} \langle \bar{\psi}_i \gamma^0 \psi_i \rangle \right] \langle \bar{\psi}_i \psi_i \rangle, \quad (B.1) $$

where

$$ \langle \bar{\psi}_i \psi_i \rangle = \frac{\nu_i m_i}{2\pi^2} \int_{\Lambda_i} \frac{p^2}{\sqrt{p^2 + m_i^2}} (n_i^+ - n_i^-) \, d|p|, \quad (B.2) $$

$$ \langle \bar{\psi}_i \gamma^0 \psi_i \rangle = \frac{\nu_i}{2\pi^2} \int_{\Lambda_i} p^2 (n_i^+ + n_i^- - 1) \, d|p| = \rho_i, \quad (B.3) $$

with

$$ n_i^\pm = \left[ e^{\beta (\sqrt{p^2 + m_i^2} - \mu_i)} + 1 \right]^{-1}, \quad (B.4) $$

$$ \mu_i^T = \mu_i - 2\left[ G_{sv}^i \langle \bar{\psi}_i \psi_i \rangle \right] \langle \bar{\psi}_i \gamma^0 \psi_i \rangle, \quad (B.5) $$

and $\Lambda_i$ being the three-momentum cutoff. The above equations are solved self-consistently as

$$ 1 + 2G_s^i \left[ 1 - \frac{G_{sv}^i}{G_s^i} \rho_i^2 \right] \frac{\nu_i}{2\pi^2} \int_0^{\Lambda_i} \frac{p^2}{\sqrt{p^2 + m_i^2}} (n_i^+ - n_i^-) \, d|p| = 0, \quad (B.6) $$

$$ \rho_i - \frac{\nu_i}{2\pi^2} \int_0^{\Lambda_i} p^2 (n_i^+ + n_i^- - 1) \, d|p| = 0. \quad (B.7) $$

Also, the pressure (2.16) is

$$ p_i(T, \mu_i) = -\left[ \langle \mathcal{H}_i^{MF} \rangle_{(T, \mu_i)} - \langle \mathcal{H}_i^{MF} \rangle_{(T=0, \mu_i=m_i(T=0))} \right] + \mu_i \langle \mathcal{N}_i \rangle + T \langle \mathcal{S}_i \rangle, \quad (B.8) $$

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where

\begin{align}
\langle \mathcal{H}^{MF}_{t}(x, \mu_i) \rangle &= \langle \bar{\psi}_i (\gamma \cdot p) \psi_i \rangle - G_s \langle \bar{\psi}_i \gamma^0 \psi_i \rangle^2 + G_\nu \langle \bar{\psi}_i \gamma^0 \gamma^0 \psi_i \rangle^2 + G_{\text{sv}} \langle \bar{\psi}_i \gamma^0 \psi_i \rangle^2 , \quad (B\,9) \\
(H^{MF}_{t}(T=0, \mu_i = m_i (T=0)) &= \langle \bar{\psi}_i (\gamma \cdot p) \psi_i \rangle - G_s \langle \bar{\psi}_i \psi_i \rangle^2 , \quad (B\,10) \\
\langle \mathcal{N}_i \rangle &= \langle \bar{\psi}_i \gamma^0 \psi_i \rangle = \rho_i , \quad (B\,11) \\
\langle \mathcal{S}_i \rangle &= -\frac{\nu_i}{2\pi^2} \int d|p| \ p^2 [n_+^i \ln n_+^i + (1-n_+^i) \ln(1-n_+^i) + n_-^i \ln n_-^i + (1-n_-^i) \ln(1-n_-^i)] , \quad (B\,12)
\end{align}

with

\begin{align}
\langle \bar{\psi}_i (\gamma \cdot p) \psi_i \rangle &= \frac{\nu_i}{2\pi^2} \int d|p| \ \frac{|p|^4}{\sqrt{p^2 + m_i^2}} (n_+^i - n_-^i) , \quad (B\,13) \\
\langle \bar{\psi}_i \psi_i \rangle &= -\frac{\nu_i m_i}{2\pi^2} \int_{p_F}^{\Lambda_i} d|p| \ \frac{p^2}{\sqrt{p^2 + m_i^2}} , \quad (B\,14) \\
\langle \bar{\psi}_i \gamma^0 \psi_i \rangle &= \rho_i (T=0) = \frac{\nu_i}{6\pi^2} p_F^1 , \quad (B\,15)
\end{align}

and $p_F^1 = \sqrt{\mu_i^2 - m_i^2}$ being the Fermi momentum, as written in Eqs. (2.12)-(2.17). In order to numerically solve the above equations, we should nondimensionalize the following finite-dimensional values as

\begin{align}
|p| \ [\text{MeV}] &\rightarrow |p|/\Lambda_i \equiv |p'| , \quad (B\,16) \\
m_i \ [\text{MeV}] &\rightarrow m_i/\Lambda_i \equiv x_i , \quad (B\,17) \\
\mu_i \ [\text{MeV}] &\rightarrow \mu_i/\Lambda_i \equiv y_i , \quad (B\,18) \\
\beta(=1/T) \ [\text{MeV}^{-1}] &\rightarrow \beta\Lambda_i (= \Lambda_i/T) \equiv t , \quad (B\,19) \\
\rho_i \ [\text{MeV}^3] &\rightarrow \rho_i/\Lambda_i^3 \equiv r_i . \quad (B\,20)
\end{align}

Thus, the self-consistent equations (B.6)-(B.7) with Eqs. (B.1)-(B.5) are recast in the nondimensionalized functions:

\begin{align}
f(x_i, y_i, t, r_i) &= 1 + \frac{\nu_i}{\pi^2} (G_s \Lambda_i^2) \left[ 1 - \frac{(G_s \Lambda_i^2) \nu_i}{(G_s \Lambda_i^2)^2} \right] I_1^i = 0 , \quad (B\,21) \\
g(x_i, y_i, t, r_i) &= r_i - \frac{\nu_i}{2\pi^2} I_2^i = 0 , \quad (B\,22) \\
I_1^i &= \int_0^1 \frac{p^2}{\sqrt{p^2 + x_i^2}} (n_+^i - n_-^i) \ d|p'| , \quad (B\,23) \\
I_2^i &= \int_0^1 p^2 (n_+^i + n_-^i - 1) \ d|p'| , \quad (B\,24)
\end{align}

where

\begin{align}
n_+^i &= \left[ e^{t(\pm\sqrt{p^2 + x_i^2} - y_i)} + 1 \right]^{-1} , \quad (B\,25) \\
y_i (\equiv \mu_i/\Lambda_i) &= y_i^1 + 2 [(G_s \Lambda_i^2) + (G_s \Lambda_i^2) z_i^2] r_i , \quad (B\,26) \\
z_i (\equiv \langle \bar{\psi}_i \gamma \psi_i \rangle / \Lambda_i^3) &= \frac{\nu_i x_i}{2\pi^2} I_1^i . \quad (B\,27)
\end{align}

\footnote{The zero-temperature fermion number distribution functions $n_+^i (T=0)$ become to the Heaviside step function $n_+^i = \theta (\mu_i - \sqrt{p^2 + m_i^2})$ and $n_-^i = 1$.}
In addition, the nondimensionalized pressure \( \overline{p}_i(=p_i/\Lambda_i^4) \) is rewritten as

\[
\overline{p}_i = - [H_i - H^0_i] + y_i r_i + s_i/t , \tag{B-28}
\]

where

\[
H_i \equiv \langle H^{MF}_{i}(T,\mu) / \Lambda_i^4 \rangle = v_i - (G_s^i \Lambda_i^2) z_i^2 + (G_v^i \Lambda_i^2) r_i^2 + (G_{sv}^i \Lambda_i^8) r_i^2 z_i^2 , \tag{B-29}
\]

\[
v_i \equiv \langle \psi_i (\mathbf{r} \cdot \mathbf{p}) \psi_i \rangle / \Lambda_i^4 = \frac{\nu_i}{2\pi^2} I^i_3 , \tag{B-30}
\]

\[
s_i \equiv \langle S_i \rangle / \Lambda_i^4 = -\frac{\nu_i}{2\pi^2} I^i_4 , \tag{B-31}
\]

\[
I^i_3 = \int_0^1 \frac{p'^4}{\sqrt{p'^2 + x^2}} (n^i_+ - n^i_-) \, d|p'| , \tag{B-32}
\]

\[
I^i_4 = \int_0^1 p'^2 [n^i_+ \ln n^i_+ + (1 - n^i_+) \ln (1-n^i_+) + n^i_- \ln n^i_- + (1 - n^i_-) \ln (1 - n^i_-)] \, d|p'| , \tag{B-33}
\]

\[
H^0_i \equiv \langle H^{MF}_{i}(T=0, \mu_i=m_i(T=0)) / \Lambda_i^4 \rangle = v^0_i - (G_s^i \Lambda_i^2) (z^0_i)^2 , \tag{B-34}
\]

\[
v^0_i \equiv \langle \psi_i (\mathbf{r} \cdot \mathbf{p}) \psi_i \rangle (\mu_i=m_i(T=0)) / \Lambda_i^4 = \frac{\nu_i}{2\pi^2} I^i_3 , \tag{B-35}
\]

\[
z^0_i \equiv \langle \psi_i \psi_i \rangle (\mu_i=m_i(T=0)) / \Lambda_i^4 = \frac{\nu_i}{2\pi^2} I^i_4 , \tag{B-36}
\]

\[
\overline{I}_1 = -\int_0^1 \frac{p'^2}{\sqrt{p'^2 + x^2}} (\pi^i_+ - \pi^i_-) \, d|p'| , \tag{B-37}
\]

\[
= -\frac{1}{2} \left[ \sqrt{1+x^2} - x \ln \left( 1 + \sqrt{1+x^2} \right) + x^2 \ln(x) \right]_{x=x_0^i} , \tag{B-38}
\]

\[
\overline{I}_3 = -\int_0^1 \frac{p'^4}{\sqrt{p'^2 + x^2}} (\pi^i_+ - \pi^i_-) \, d|p'| , \tag{B-39}
\]

\[
= -\frac{1}{8} \left[ (2 - 3x^2) \sqrt{1+x^2} + 3x^4 \ln(1 + \sqrt{1+x^2}) + 3x^4 \ln(x) \right]_{x=x_0^i} , \tag{B-40}
\]

with

\[
\pi^i_+ = n^i_+(T=0) = \theta(\mu_i - \sqrt{p^2 + m_i^2})
\]

\[
= \left\{ \begin{array}{ll}
1 & (p < \sqrt{\mu_i^2 - m_i^2} \equiv p_F) \\
0 & (p > \sqrt{\mu_i^2 - m_i^2} \equiv p_F)
\end{array} \right. , \tag{B-41}
\]

\[
\pi^i_- = n^i_-(T=0) = 1 . \tag{B-42}
\]

Here, we fixed the vacuum masses of quark matter and nuclear matter as \( m_q(\rho_v=0, T=0) = 313 \) MeV and \( m_N(\rho_v=0, T=0) = 939 \) MeV (see [2.1.2] in Chapter 2), so that \( x^q_0 = 313 / \Lambda_q \) and \( x^N_0 = 939 / \Lambda_N \), and thus \( H^0_q = -0.155107 \) and \( H^0_N = -0.0956302 \). The values of cutoff \( \Lambda_i \) and that of nondimensionalized model-parameters, \( G_s^i \Lambda_i^2 \), \( G_v^i \Lambda_i^2 \), and \( G_{sv}^i \Lambda_i^8 \), are summarized in Table I and II for \( i = q \) and \( i = N \). Consequently, the pressure \( (B-28) \) of quark matter \( (i = q) \) and that of nuclear matter \( (i = N) \) are, respectively,

\[
\overline{p}_q = -(H_q + 0.155107) + y_q r_q + s_q / t , \tag{B-43}
\]

\[
\overline{p}_N = -(H_N + 0.0956302) + y_N r_N + s_N / t . \tag{B-44}
\]
B.2 Numerical root finding

In order to find solutions or roots of the self-consistent equations (B.21) and (B.22), we first plug the fixed $T$ and the given $\mu_i^t$ into the equations:

\[
\begin{align*}
  f(x_i, r_i)|_{y_i^t=y_i^{\text{con}, t=t_{\text{Fixed}}}} &= 1 + \frac{\nu_i}{\pi^2} (G_s^i \Lambda_c^2 - \frac{(G_s^i \Lambda_c^8)}{(G_s^i \Lambda_c^2)^2}) \int_{t_0}^1 I_1^i \, dt = 0, \quad (B.45) \\
  g(x_i, r_i, y_i^t) &= r_i - \frac{\nu_i}{2\pi^2} I_2^i \, dt = 0, \quad (B.46) \\
  I_1^i &= \int_0^1 \frac{p^2}{\sqrt{p^2 + x_i^2}} (n_i^+ - n_i^-) \, dp \, | \, , \quad (B.47) \\
  I_2^i &= \int_0^1 p^2 (n_i^+ + n_i^- - 1) \, dp \, | \, , \quad (B.48) \\
  n_i^\pm &= e^{\text{Fixed}(\pm \sqrt{p^2 - x_i^2} - y_i^{\text{con}})} + 1 \, | -1 . \quad (B.49)
\end{align*}
\]

Next, we can find roots of Eq. (B.21) and (B.22) by using the recurrence relations for Newton-Raphson method\footnote{For the details for the Newton-Raphson method, see the next section.}

\[
\begin{pmatrix}
  x_i^{(k+1)} \\
  r_i^{(k+1)}
\end{pmatrix} = \begin{pmatrix}
  x_i^{(k)} \\
  r_i^{(k)}
\end{pmatrix} - J^{-1}(x) \begin{pmatrix}
  f(x_i^{(k)}, r_i^{(k)}) \\
  g(x_i^{(k)}, r_i^{(k)})
\end{pmatrix}, \quad (k = 1, 2, \cdots) \quad (B.50)
\]

where

\[
J^{-1}(x) = \frac{1}{J_{11}J_{22} - J_{12}J_{21}} \begin{pmatrix}
  J_{22} & -J_{12} \\
  -J_{21} & J_{11}
\end{pmatrix}, \quad (B.51)
\]

with

\[
\begin{align*}
  J_{11} &= \frac{\partial f(x_i, r_i)}{\partial x_i} = \frac{\nu_i}{\pi^2} (G_s^i \Lambda_c^2) \frac{\partial I_1^i}{\partial x_i}, \quad (B.52) \\
  J_{12} &= \frac{\partial f(x_i, r_i)}{\partial r_i} = - \frac{2\nu_i r_i}{\pi^2} (G_s^i \Lambda_c^8) I_1^i, \quad (B.53) \\
  J_{21} &= \frac{\partial g(x_i, r_i)}{\partial x_i} = - \frac{\nu_i}{2} \frac{\partial I_2^i}{\partial x_i}, \quad (B.54) \\
  J_{22} &= \frac{\partial g(x_i, r_i)}{\partial r_i} = 1. \quad (B.55)
\end{align*}
\]

Here, the partial derivatives are

\[
\begin{align*}
  \frac{\partial I_1^i}{\partial x_i} &= \int_0^1 \frac{\partial}{\partial x_i} \frac{p^2}{\sqrt{p^2 + x_i^2}} (n_i^+ - n_i^-) \, dp \, | , \quad (B.56) \\
  \frac{\partial I_2^i}{\partial x_i} &= \int_0^1 \frac{\partial}{\partial x_i} p^2 (n_i^+ + n_i^- - 1) \, dp \, | , \quad (B.57)
\end{align*}
\]
where the integrands are

\[
\frac{\partial}{\partial x_i} \left[ \frac{p_i'^2}{\sqrt{p^2 + x_i^2}} (n^i_+ - n^i_-) \right]
\]

\[
= p_i'^2 x_i \left[ \frac{\left( e^{\text{Fixed}(\sqrt{p^2 + x_i^2} + y_i^e_Given) + 1} - 1 \right)}{\left( p^2 + x_i^2 \right)^{3/2}} - \frac{e^{\text{Fixed}(\sqrt{p^2 + x_i^2} + y_i^e_Given) \left[ (e^{\text{Fixed}(\sqrt{p^2 + x_i^2} + y_i^e_Given) + 1} - 1 \right)}{p^2 + x_i^2} \right] ,
\]

\[
\frac{\partial}{\partial x_i} \left[ p_i'^2 (n^i_+ + n^i_- - 1) \right]
\]

\[
= - \frac{p_i'^2 t^{\text{Fixed}} x_i (e^{2 t^{\text{Fixed}} y_i^e_Given - 1})(e^{2 t^{\text{Fixed}} \sqrt{p^2 + x_i^2} - 1} - 1) e^{\text{Fixed}(\sqrt{p^2 + x_i^2} + y_i^e_Given) \left[ (e^{\text{Fixed}(\sqrt{p^2 + x_i^2} + y_i^e_Given) + 1} - 1 \right)}{\sqrt{p^2 + x_i^2} e^{\text{Fixed}(\sqrt{p^2 + x_i^2} + y_i^e_Given) + 1}^2} .
\]

(B·38)

(B·39)

In this way, the roots \( x_i \) and \( r_i \) are obtained, and thereby we can get the value \( y_i \) by inversely solving the equation (B·26) with the fixed \( T \), the given \( \mu_i^r \), and the obtained \( x_i \) and \( r_i \). If we have the five values \( (x_i, y_i, t, r_i, y_i) \), we can derive the value of \( p_i \) by substituting the five values into Eq. (B·28). Therefore, we completely solve the NJL gap equation (2·3) and the pressure (2·1).

B·3 Numerical methods

In this thesis, Newton-Raphson method, also called Newton Method, is used to solve the NJL gap equation. This method is one of the well-known technique for solving equations numerically, based on the simple idea of linear approximation. Also, we used Simpson’s rule to get the values of integral numerically. In this section, we briefly introduce the Newton-Raphson method and Simpson’s rule. In addition, we also present Cooling method known as the optimal gradient method. For more details of the numerical techniques such as Newton-Raphson method, see e.g., Ref. 17, 35 in which also the numerical codes in C/C++ outlined.

B·3.1 The Newton-Raphson iteration

As a estimate of \( r \), let \( x_0 \) be a first guess, and let \( h \) be a displacement between the true root \( r \) and the estimate \( x_0 \). If \( h \) is small, the liner approximation is available, i.e.,

\[
0 = f(x_0 + h) \approx f(x_0) + hf'(x_0) .
\]

Thus, \( h \approx -f(x_0)/f'(x_0) \) if \( f'(x_0) \neq 0 \), so that \( r = x_0 + h \approx x_0 - f(x_0)/f'(x_0) \). This leads to a new estimate \( x_1 \) of \( r \), i.e., \( x_1 = x_0 - f(x_0)/f'(x_0) \). In the same way as \( x_1 \) obtained from \( x_0 \), the next estimate \( x_2 \) with \( x_1 \) is given by \( x_2 = x_1 - f(x_1)/f'(x_1) \). Therefore, the estimate \( x_n \) is derived from the previous estimate \( x_{n-1} \) as

\[
x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} .
\]

(B·60)

By continuing this procedure, the estimate value approached to the true root.
Next, we consider the multi-dimensional case for Newton-Raphson method as

\[
F(x) = \begin{bmatrix}
f_1(x_1, x_2, \ldots, x_n) \\
f_2(x_1, x_2, \ldots, x_n) \\
\vdots \\
f_n(x_1, x_2, \ldots, x_n)
\end{bmatrix} = \begin{bmatrix}
f_1(x) \\
f_2(x) \\
\vdots \\
f_n(x)
\end{bmatrix} = 0 . \tag{B 61}
\]

In the vicinity of \( x \), we can expand \( F(x) \) in Taylor series:

\[
F(x + \delta x) = F(x) + F'(x)\delta x + O(\delta x^2) , \tag{B 62}
\]

where

\[
F'(x)\delta x = \frac{\partial f_i(x)}{\partial x_j} \delta x_j , \quad (i,j = 1,2,\ldots,n) \equiv J . \tag{B 63}
\]

Here, Jacobian matrix \( J \) is defined by

\[
J = J_{ij} = \frac{\partial f_i(x)}{\partial x_j} = \begin{bmatrix}
\frac{\partial f_1(x)}{\partial x_1} & \ldots & \frac{\partial f_1(x)}{\partial x_n} \\
\frac{\partial f_2(x)}{\partial x_1} & \ldots & \frac{\partial f_2(x)}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_n(x)}{\partial x_1} & \ldots & \frac{\partial f_n(x)}{\partial x_n}
\end{bmatrix} . \tag{B 64}
\]

By linearizing Eq. (B 62), i.e., neglecting more than the second-order term \( O(\delta x^2) \), and also by setting \( F(x + \delta x) = 0 \), the following relation is obtained:

\[
\delta x = -J^{-1} \cdot F(x) . \tag{B 65}
\]

Therefore, we get the recurrence relation for Newton-Raphson method:

\[
x^{(k+1)} = x^{(k)} - J^{-1} \cdot F(x^{(k)}) . \tag{B 66}
\]

The above process is iterated to convergence. Consequently, we can find the true roots.

For instance, we take into account the two-dimensional case for Newton-Raphson method. There are two nonlinear self-consistent equations with roots \( x \) and \( y \):

\[
f(x, y) = 0 , \tag{B 67}
\]
\[
g(x, y) = 0 . \tag{B 68}
\]

The matrix elements of Jacobian \( J_{ij} \) are

\[
J_{11} = \frac{\partial f(x, y)}{\partial x} , \tag{B 69}
\]
\[
J_{12} = \frac{\partial g(x, y)}{\partial y} , \tag{B 70}
\]
\[
J_{21} = \frac{\partial f(x, y)}{\partial x} , \tag{B 71}
\]
\[
J_{22} = \frac{\partial g(x, y)}{\partial y} . \tag{B 72}
\]
In this case, the inverse matrix $J^{-1}$ can be easily calculated as
\[
J^{-1} = \frac{1}{J_{11}J_{22} - J_{12}J_{21}} \begin{bmatrix}
J_{22} & -J_{12} \\
-J_{21} & J_{11}
\end{bmatrix}.
\] (B.73)

Thus, from the Eq. (B.65),
\[
\begin{bmatrix}
\delta x \\
\delta y
\end{bmatrix} = -\frac{1}{J_{11}J_{22} - J_{12}J_{21}} \begin{bmatrix}
J_{22} & -J_{12} \\
-J_{21} & J_{11}
\end{bmatrix} \begin{bmatrix}
f(x, y) \\
g(x, y)
\end{bmatrix}
= \begin{bmatrix}
-J_22f(x, y) - J_12g(x, y) / \det(J) \\
J_21f(x, y) - J_11g(x, y) / \det(J)
\end{bmatrix}.
\] (B.74)

As a result, we can get the recurrence relations:
\[
x^{(k+1)} = x^{(k)} - \delta x,
\] (B.75)
\[
y^{(k+1)} = y^{(k)} - \delta y.
\] (B.76)

Here, the convergence criterion with $\epsilon$ is
\[
|\delta x| < \epsilon, \quad |\delta y| < \epsilon \quad \text{or} \quad \sqrt{|\delta x|^2 + |\delta y|^2} < \epsilon.
\] (B.77)

### B.3.2 Simpson’s rule

As a numerical method for calculating an approximate value for a definite integral, Simpson’s rule with quadratic polynomials is often used. In this integral method, the integral interval $[a, b]$ is divided into $2n$ equal parts, i.e., $h = (b - a)/2n$, and then $x_m = a + mh$ is assumed. The function $f(x)$ can be approximated to the function $g(x)$ with the aid of a quadratic function passing through the three points such as $(x_0, f(x_0)), (x_1, f(x_1)),$ and $(x_2, f(x_2))$, in which Lagrange interpolation method\(^3\) is used:
\[
g(x) = f(x_0) \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + f(x_1) \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} + f(x_2) \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}.
\] (B.78)

By using the integral of $g(x)$, we can approximate the integral of $f(x)$ in the interval $[x_0, x_2]$ to
\[
\int_{x_0}^{x_2} f(x) dx \approx \int_{x_0}^{x_2} g(x) dx = \frac{h}{3} [f(x_0) + 4f(x_1) + f(x_2)].
\] (B.79)

As a consequence, we obtain Simpson’s rule formula in the integral interval $[a, b]$:
\[
\int_{a}^{b} f(x) dx \approx \frac{h}{3} \left[ f(x_0) + 2 \left( f(x_2) + \cdots + f(x_{2n-2}) \right) + 4 \left( f(x_1) + \cdots + f(x_{2n-1}) \right) + f(x_{2n}) \right].
\] (B.80)

\(^3\) Lagrange interpolation method is a technique for interpolating with an $n$th-order polynomial passing through the given $[n+1]$ points such as $(x_m, y_m)$ ($m = 0, 1, \cdots, n$).
B.3.3 Cooling method

In nuclear structure studies, in order to describe the ground state, a variational method known as the frictional cooling method, also called cooling method, which has been used to perform the variational energy calculation in condensed matter physics, is applied to the nuclear wave function in antisymmetrized molecular dynamics (AMD) [50]. The cooling method is a powerful technique for calculating the variational energy with multi-dimensional parameters. By using this method, each position of the single-particle wave packets can be chosen variationally, namely, the optimal solution with minimum energy is determined automatically.

Here, we introduce the cooling method as one of the root finding method such as Newton method, and also present its application to algebraic equations [50]. In the cooling method, calculating for finding the minimal point is linked to the time-evolution of the simultaneous differential equation. Thus, it is possible to approach to the matrix eigenvalue problems or the algebraic equations numerically and variationally.

As for the cooling method, we first give a brief introduce of the concept. Let $H(q_i, p_i)$ be Hamiltonian as a function of the position $q_i$ and the momentum $p_i$ where $q_i=(q_1, q_2, \cdots, q_n)$ and $p_i=(p_1, p_2, \cdots, p_n)$. For simplicity, we consider here the two-dimensional case with one-dimensional values, i.e., $H(q_1, p_1)$, and find its local minimum energy. Let $\tau$ be a virtual time which is a moving time of a given point $P(q_1, p_1)$ in the potential energy $H(q_1, p_1)$. In order to search the minimal point of potential $H(q_1, p_1)$ with the point $P(q_1, p_1)$, we associate the moving velocity of $P(q_1, p_1)$ with the gradient of $H(q_1, p_1)$:

$$\frac{dq_1}{d\tau} = -\frac{\partial H}{\partial q_1}, \quad \frac{dp_1}{d\tau} = -\frac{\partial H}{\partial p_1}.$$  \hspace{1cm} (B.81)

Therefore, the point $P(q_1, p_1)$ has a downward velocity in which $P(q_1, p_1)$ gets down a slope of the potential. Here, the virtual time rate of change of the potential energy $H(q_1, p_1)$ becomes always negative:

$$\dot{H}(q_1, p_1) = \frac{dq_1}{d\tau} \frac{\partial H}{\partial q_1} + \frac{dp_1}{d\tau} \frac{\partial H}{\partial p_1} = -\left(\frac{\partial H}{\partial q_1}\right)^2 - \left(\frac{\partial H}{\partial p_1}\right)^2 < 0.$$  \hspace{1cm} (B.82)

Accordingly, it is possible to find a local minimum point of $H(q_1, p_1)$ in the given sufficient time $t$. In this way, by solving the equation of $P(q_1, p_1)$ in Eq. (B.81), we can find a local minimum of $H(q_1, p_1)$. This treatment corresponds to the operation for searching the minimal energy. If we assume the values $(q_1, p_1)$ as canonical ones with real time $t$, Eq. (B.81) becomes like the Hamilton’s canonical equations of motion:

$$\frac{dq_1}{dt} = -\frac{\partial H}{\partial p_1}, \quad \frac{dp_1}{dt} = -\frac{\partial H}{\partial q_1}.$$  \hspace{1cm} (B.83)

Therefore, in the canonical form, calculating the ground energy is associated with a calculation for investigating the dynamics of the system. In the case with constraint conditions, it is also available

---

4 In this method, the nuclear ground state is obtained by cooling the energy of the system, introducing the frictional cooling equation including the frictional term.

5 Actually, it is sufficient to find a local minimum point with finite time. Also, some may be settled down at a not minimum point.
by using Lagrange multipliers method:

\[ \mathcal{H}(q_1, p_1) = H + \mu \left( N(q_1, p_1) - C_0 \right) . \]  

Let us consider the problem on finding of local minimum of \( H(q_1, p_1) \), keeping a given function \( N(q_1, p_1) \) constant with \( C_0 \). Plugging the values of \( q_1 \) and \( p_1 \) derived by the virtual time evolution into \( H(q_1, p_1) \), we get the local minimum of \( H(q_1, p_1) \) under the condition \( N(q_1, p_1) - C_0 \) after we solve the following simultaneous differential equation with Lagrange multipliers method:

\[ \frac{dq_1}{d\tau} = -\frac{\partial \mathcal{H}}{\partial q_1}, \quad \frac{dp_1}{d\tau} = -\frac{\partial \mathcal{H}}{\partial p_1}, \quad \frac{d\mu}{d\tau} = -\frac{\partial \mathcal{H}}{\partial \mu} . \]  

Now, we apply the cooling method to algebraic equations. We first take into account the one-dimensional case. In order to find \( x \) satisfies \( f(x) = 0 \) in which the solution \( x \) intersect with the \( x \) axial, the potentialization of \( f(x) \) is needed here. The potentialization means the squared \( f(x) \), i.e., \( V(x) = f^2(x) \) where \( V(x) \) represents the potentialization. By solving the equation of motion after the potentialization, the minimal point with \( V(x) = 0 \) is obtained:

\[ \dot{x} = -\frac{\partial V}{\partial x} = -2f(x) \cdot \frac{\partial f}{\partial x} . \]  

Here, Eq. (B.86) is numerically solved using e.g., Runge-Kutta method.\footnote{For instance, we take into account the problem: \( y' = f(t, y) \) and \( y(t_0) = \alpha \). Let \( h \) be the time step size and \( t_i = t_0 + ih \). Then, the following formula finds an approximate solution:

\[ \omega_0 = \alpha , \]  

\[ k_1 = hf(t_i, \omega_i) , \]  

\[ k_2 = hf(t_i + h/2, \omega_i + k_1/2) , \]  

\[ k_3 = hf(t_i + h/2, \omega_i + k_2/2) , \]  

\[ k_4 = hf(t_i + h, \omega_i + k_3) , \]  

\[ \omega_{i+1} = \omega_i + (k_1 + 2k_2 + 2k_3 + k_4) / 6 , \]  

in which \( \omega_i \approx y(t_i) \). For more details on Runge-Kutta method, see Ref. [7] [8].}

In the two-dimensional case, in order to find \( x \) and \( y \) satisfies \( f(x, y) = 0 \) and \( g(x, y) = 0 \), we need the potentialization as \( V(x, y) = f^2(x, y) + g^2(x, y) \). The equations of motion for a given point \( P(x, y) \) in the two-dimensional potential are

\[ \dot{x} = -\frac{\partial V}{\partial x} = -2\frac{\partial f}{\partial x} - 2\frac{\partial g}{\partial x} , \]  

\[ \dot{y} = -\frac{\partial V}{\partial y} = -2\frac{\partial f}{\partial y} - 2\frac{\partial g}{\partial y} . \]  

Here, the stable point in the potential becomes a solution or a root to solve. It should be noted that there are no matrix calculations as in Newton method, while a check of \( V(x, y) = 0 \) in the point where \( P(x, y) \) settled down is needed.
Similarly, in the \( n \)-dimensional case, the corresponding equations of motion are

\[
\begin{align*}
    f_1(x_1, x_2, \cdots, x_n) &= 0, \\
    f_2(x_1, x_2, \cdots, x_n) &= 0, \\
    \vdots \\
    f_n(x_1, x_2, \cdots, x_n) &= 0,
\end{align*}
\]

and its potentialization is

\[
V(x_1, x_2, \cdots, x_n) = \sum_{k=1}^{n} f_k^2(x_1, x_2, \cdots, x_n). \tag{B-96}
\]

Here, the equations of motion for \( P(x_1, x_2, \cdots, x_n) \) are

\[
\begin{align*}
    x_1' &= -\frac{\partial V}{\partial x_1} = -2\sum_{k=1}^{n} f_k \frac{\partial f_k}{\partial x_1}, \\
    x_2' &= -\frac{\partial V}{\partial x_2} = -2\sum_{k=1}^{n} f_k \frac{\partial f_k}{\partial x_2}, \\
    \vdots \\
    x_n' &= -\frac{\partial V}{\partial x_n} = -2\sum_{k=1}^{n} f_k \frac{\partial f_k}{\partial x_n}, \tag{B-99}
\end{align*}
\]

where the stable point becomes a solution to solve.

Finally, we introduce an application to the matrix eigenvalue problems. For simplicity, we are concerned here with a square matrix, i.e., \( 2 \times 2 \) matrix. For example, let us consider the following matrix:

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{B-100}
\]

Now, we solve the eigenvalue \( E \) and its eigenvector \( v = (x, y)^t \) for the matrix \( A \) using the cooling method:

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = E \begin{pmatrix} x \\ y \end{pmatrix}, \tag{B-101}
\]

with matrix elements \( a, b, c, \) and \( d \). Thus, Eq. (B-101) in the simultaneous equation form can be written as

\[
\begin{align*}
    f(x, y, E) &= (a - E)x + by = 0, \tag{B-102} \\
    g(x, y, E) &= cx + (d - E)y = 0. \tag{B-103}
\end{align*}
\]

By potentializing the above equations, we obtain

\[
V(x, y, E) = f^2(x, y, E) + g^2(x, y, E) = 0, \tag{B-104}
\]

and consequently get the following equations of motion:

\[
\frac{dx}{d\tau} = -\frac{\partial V}{\partial x}, \quad \frac{dy}{d\tau} = -\frac{\partial V}{\partial y}, \quad \frac{dE}{d\tau} = -\frac{\partial V}{\partial E}. \tag{B-105}
\]
Therefore, setting initial values to \( x = x_0, y = y_0, E = E_0 \), and providing time evolution, we can find the solution. However, one may return the trivial solution such as \( x = y = 0 \), so that we impose the normalization condition as \( x^2 + y^2 = 1 \) in order to find the real root. Thus, we consider \( h(x, y, E) = x^2 + y^2 - 1 = 0 \) in addition to Eq. (B.102) and (B.103), and hence the potentialization is as follows:

\[
F(x, y, E) = f^2(x, y, E) + g^2(x, y, E) + h^2(x, y, E) = 0 \tag{B.106}
\]

in which the solution does not choose the point \((x, y) = (0, 0)\) due to the fourth-dimensional potential with \( h(x, y, E) \). As a result,

\[
\begin{align*}
\frac{dx}{d\tau} &= -\frac{\partial F}{\partial x} , \\
\frac{dy}{d\tau} &= -\frac{\partial F}{\partial y} , \\
\frac{dE}{d\tau} &= -\frac{\partial F}{\partial E} .
\end{align*} \tag{B.107}
\]

From the above equations, the eigenvalue \( E \) and its eigenvector \( \mathbf{v}_c = (x_c, y_c) \) are obtained, while the solution \( E \) depends on the initial values \((x_0, y_0, E_0)\).

### B.4 Numerical Results

In this section, more details on the numerical results are outlined. By solving the Eq. (B.11) with numerical methods in the previous section, the NJL gap solutions are obtained.

For the quark matter \((i = q)\) with the parameter set of Table II in (B.12) we present the figures on the dynamical mass-chemical potential \((m_q-\mu_q)\) relation, the number density-chemical potential \((n_q-\mu_q)\) relation, and the pressure-chemical potential \((p_q-\mu_q)\) relation, taking \( G_{uu}A_q^8 = -68.4 \) as an example for the free parameter.

In Fig. B-1 and B-2, the \( m_q-\mu_q \) relation, i.e., the gap solutions as a function of the quark chemical potential, are depicted. The vertical axis and the horizontal one represent the dynamical quark mass and the quark chemical potential, respectively. At zero temperature, the vacuum value of \( m_q \) is obtained for \( \mu_q = 313 \) MeV. Figure B-2 covers \( \mu_q \) from 0 to 330 MeV, while the scope of \( \mu_q \) in Fig. B-1 is from 313 to 330 MeV. As seen in Fig B-1 at \( T = 0 \sim 25 \) MeV, the gap equation has multiple solutions in a certain region.

In Fig. B-3 and B-4, the relation between the quark number density and the quark chemical potential \((\rho_q-\mu_q \) relation) are depicted. The horizontal axis is the quark chemical potential. The vertical axis denotes the quark number density over 0.17. Here, 0.17 represents the normal nuclear density, and its unit is the negative cube of femtometer, i.e., \( \text{fm}^{-3} \). By using the conversion factor \( \hbar c \approx 197 \text{ MeVfm} \) with \( c \) being the speed of light and \( \hbar \) being the reduced Planck constant or Dirac’s constant, we can convert energy to length and vice versa. In natural units, \( \hbar \) and \( c \) become equal to one, i.e., \( \hbar = c = 1 \). Therefore, \( 0.17 \text{ fm}^{-3} = 0.17 \times 197^3 \text{ MeV}^3 \). Then, the quark number density divided by the normal nuclear density, \( \rho_q/(0.17 \times 197^3) \), becomes dimensionless. In Fig. B-3, in order to make the unphysical region more visible, we set the scope of the quark chemical potential from 313 to 330 MeV, while Figure B-4 covers \( \mu_q \) from 0 to 330 MeV.
**Fig. B-1:** The gap solutions ($m_q\mu_q$ relation) at $T = 0 \sim 40$ MeV. The symbol $u_q$ represents $\mu_q$ in the caption of the horizontal axis. The scope of the quark chemical potential is from 313 MeV to 330 MeV.

**Fig. B-2:** The gap solutions ($m_q\mu_q$ relation) at $T = 50 \sim 190$ MeV. The symbol $u_q$ represents $\mu_q$ in the caption of the horizontal axis. The quark chemical potential covers from 0 to 330 MeV.
**Fig. B-3:** The relation between the quark number density and the quark chemical potential, i.e., the $\rho_q^{\mu_q}$ relation at $T = 0 \sim 40$ MeV. In each of the axis captions, the symbols $\rho_q$ and $\mu_q$ represent $\rho_q$ and $\mu_q$, respectively. The scope of the quark chemical potential is from 313 to 330 MeV. In the vertical axis, the quark number density is divided by the normal nuclear density, 0.17 fm$^{-3}$.

**Fig. B-4:** The relation between the quark number density and the quark chemical potential, i.e., the $\rho_q^{\mu_q}$ relation at $T = 50 \sim 190$ MeV. In each of the axis captions, the symbols $\rho_q$ and $\mu_q$ represent $\rho_q$ and $\mu_q$, respectively. The scope of the quark chemical potential is from 0 to 330 MeV. In the vertical axis, the quark number density is divided by the normal nuclear density, 0.17 fm$^{-3}$. 
Fig. B-5: Upper left-hand side panel: The quark pressure $p_q$ as a function of the quark chemical potential $\mu_q$ with gap solutions at $T = 0 \sim 40$ MeV. Upper right-hand side panel: The quark pressure $p_q$ as a function of the quark chemical potential $\mu_q$ with gap solutions at $T = 50 \sim 190$ MeV. Bottom panel: The quark pressures with gap solutions and massless ones as a function of the quark chemical potential at $T = 0 \sim 35$ MeV. The curves represent the pressure using the gap solution, while the lines denote the pressure using the massless solution. The crossing point stands for the chiral phase transition point.

The upper two panels in Fig. B-5, the $p_q$-$\mu_q$ relation are depicted using only gap solutions (without massless solutions). The vertical axis and the horizontal one are the quark pressure and the quark chemical potential, respectively. Here, the symbols $p_q$ or $P_q$ and $\mu_q$ represent $p_q$ and $\mu_q$, respectively. The upper right-hand side panel in Fig. B-5 covers $\mu_q = 0 \sim 330$ MeV and $p_q = 0 \sim 200$ MeV/fm$^{-3}$, while the upper left-side one $\mu_q = 313 \sim 330$ MeV and $p_q = -4 \sim 16$ MeV/fm$^{-3}$. On the other hand, the bottom panel in Fig. B-5, the quark pressures with gap solutions and massless ones are depicted as a function of the quark chemical potential within $T = 0 \sim 35$ MeV and $\mu_q = 910 \sim 990$ MeV. The chiral phase transition occurs at the crossing point since the solution with constituent (dynamical) quark mass takes to massless solution branch where the chiral symmetry is restored.
**Fig. B-6:** The $m_N - \mu_N$ relation at $T = 0 \sim 190$ MeV. In each of the axis captions, the symbols $m_N$ and $\mu_N$ represent $m_N$ and $\mu_N$, respectively.

**Fig. B-7:** The relation between the nuclear number density and the nuclear chemical potential, i.e., the $\rho_N - \mu_N$ relation at $T = 0 \sim 190$ MeV. In each of the axis captions, the symbols $\rho_N$ and $\mu_N$ denote $\rho_N$ and $\mu_N$, respectively. The nuclear number density is divided by the normal nuclear density, $0.17$ fm$^{-3}$. 
Fig. B-8: The nuclear pressure is depicted \( p_N \) as a function of the nuclear chemical potential \( u_N \) at \( T = 0 \sim 190 \text{ MeV} \).

As for the nuclear matter \((i = N)\) with the parameter set of Table I in [2.1.2] we show the figures on the mass-chemical potential \((m_N-\mu_N)\) relation, the number density-chemical potential \((\rho_N-\mu_N)\) relation, and the pressure-chemical potential \((p_N-\mu_N)\) relation.

In Fig B-6, the \( m_N-\mu_N \) relation are depicted. The vertical axis and the horizontal one represent the nuclear mass and the nuclear chemical potential, respectively. At zero temperature, the vacuum value of \( m_N \) is obtained for \( \mu_q = 939 \text{ MeV} \). Figure B-6 covers \( \mu_N \) from 0 to 1200 MeV. As seen in Fig B-6 at low temperatures, there are multiple solutions in a certain region.

In Fig B-7, the relation between the nuclear number density and the nuclear chemical potential \((\rho_N-\mu_N \text{ relation})\) are depicted. The horizontal axis is the nuclear chemical potential and its scope is \( \mu_N = 0 \sim 1200 \text{ MeV} \). The vertical axis denotes the nuclear number density divided by the normal nuclear density 0.17. Here, \( \rho_q/(0.17 \times 197^3) \) under the nondimensionalization.
Appendix C

Additional Terms in Quark NJL Model

C.1 Quark-pair interaction

In cold and dense QCD regime, it is believed that the color superconducting phase, since the perturbative ground state of cold and dense quark matter becomes unstable if there exist an attractive interaction between quarks. For instance, the so-called 2SC phases characterized by the quark pairing in two flavors, i.e., up and down quarks. Here, it is interesting to investigate the interplay of the chiral and color-superconducting phase transitions. Thus, by adding the following term with $G_c$ being the coupling constant of the quark-pair interaction \( \Xi \) to Eq. (2.1), we consider the two-flavor case including the quark-pair interaction derived by Fierz transformations from the original NJL Lagrangian:

\[
\mathcal{L}_c^q = G_c^q \sum_{\alpha=2,5,7} \left[ (\bar{\psi}_q i\gamma_5 \tau_2 \lambda_\alpha \psi_q^C)(\bar{\psi}_q^C i\gamma_5 \tau_2 \lambda_\alpha \psi_q) + (\bar{\psi}_q \tau_2 \lambda_\alpha \psi_q^C)(\bar{\psi}_q^C \tau_2 \lambda_\alpha \psi_q) \right],
\]

where $\psi_q^C = C\bar{\psi}_q^T$ and $\bar{\psi}_q^C = \psi_q^T C$ with $C = i\gamma^2 \gamma^0$, being the charge conjugation operator. Here, $T$ denotes the transposition operation, not temperature. Also, $\tau_2$ and $\lambda_\alpha$ are the second component of the Pauli matrices representing the isospin $SU(2)$-generator and the antisymmetric Gell-Mann matrices representing the color $SU(3)$-generator, respectively. Under the mean field approximation, the Lagrangian density is written as

\[
\mathcal{L}_q^{MF} = \bar{\psi}_q \left( i\gamma^\mu \partial_\mu - m_q - \bar{\mu}_q \gamma^0 \right) \psi_q - \frac{1}{2} \sum_{\alpha=2,5,7} (\Delta^*_\alpha \bar{\psi}_q i\gamma_5 \tau_2 \lambda_\alpha \psi_q + \text{h.c.}) + U_q,
\]

\[
U_q \equiv -G_q^2 \langle \bar{\psi}_q \gamma_0 \psi_q \rangle^2 + G_q^2 \langle \bar{\psi}_q i\gamma_5 \tau_2 \lambda_\alpha \psi_q \rangle^2 + 3G_q^2 \langle \bar{\psi}_q \gamma_0 \psi_q \rangle^2 \langle \bar{\psi}_q i\gamma_5 \tau_2 \lambda_\alpha \psi_q \rangle^2 - \sum_{\alpha=2,5,7} G_q^2 \langle \bar{\psi}_q i\gamma_5 \tau_2 \lambda_\alpha \psi_q^C \rangle \langle \bar{\psi}_q^C i\gamma_5 \tau_2 \lambda_\alpha \psi_q \rangle,
\]

\( ^1 \)This is because of the Cooper instability that develops as a result of the formation of quark Cooper pairs around the Fermi surface.
where
\[
\begin{align*}
m_q &= -2 \left[ G^2_q - G^2_q \langle \overline{\psi}_q \gamma^0 \psi_q \rangle^2 \right] \langle \overline{\psi}_q \gamma^0 \psi_q \rangle, \\
\bar{\mu}_q &= 2 \left[ G^2_q + G^2_q \langle \overline{\psi}_q \gamma^0 \psi_q \rangle^2 \right] \langle \overline{\psi}_q \gamma^0 \psi_q \rangle, \\
\Delta_\alpha &= -2G^0_q \langle \overline{\psi}_q i\gamma_5 \tau_2 \lambda_\alpha \psi_q \rangle, \\
\Delta^*_\alpha &= -2G^0_q \langle \overline{\psi}_q i\gamma_5 \tau_2 \lambda_\alpha \psi_q^c \rangle,
\end{align*}
\]  

Here, we used Dirac representation for the Dirac gamma matrices, and the expectation values of the pseudoscalar bilinears are ignored, i.e., \( \langle \overline{\psi}_q \gamma_5 \tau_2 \lambda_\alpha \psi_q \rangle = \langle \overline{\psi}_q \gamma_5 \tau_2 \lambda_\alpha \psi_q^c \rangle = 0 \). Also, \( \text{h.c.} \) stands for the Hermitian conjugate of the preceding term, i.e., \( \Delta_\alpha \overline{\psi}_q i\gamma_5 \tau_2 \lambda_\alpha \psi_q^c \), and \( m_q \) denotes the dynamical quark mass. If we assume that all quark-pair condensates have the same expectation value in order to insure color symmetry, the expectation value \( \Delta_q \), which corresponds to the quark-pair condensation, is as follows:
\[
\begin{align*}
\Delta_\alpha &= \Delta^*_\alpha = -2G^0_q \langle \overline{\psi}_q i\gamma_5 \tau_2 \lambda_\alpha \psi_q \rangle, \\
\Delta_q &= \sum_{\alpha=2,5,7} \Delta_\alpha = 5 = \Delta_\alpha = 7.
\end{align*}
\]  

Thus, the term \( \sum_{\alpha=2,5,7} G^2_q \langle \overline{\psi}_q i\gamma_5 \tau_2 \lambda_\alpha \psi_q \rangle \langle \overline{\psi}_q i\gamma_5 \tau_2 \lambda_\alpha \psi_q^c \rangle \) in Eq. (C.2) is recast into \( 3\Delta_q^2 (4G^2_q)^{-1} \). Here, it means the existence of the 2SC phase if \( \Delta_q \neq 0 \). The mean field Hamiltonian density with quark chemical potential \( \mu_q \) is obtained as
\[
\begin{align*}
\mathcal{H}'_q &= \mathcal{H}'_q \mathcal{M}F - \mu_q \overline{\psi}_q \psi_q \\
&= -i\overline{\psi}_q \gamma \cdot \nabla \psi_q + m_q \overline{\psi}_q \psi_q - \mu_q \overline{\psi}_q \gamma^0 \psi_q \\
&\quad + \frac{1}{2} \sum_{\alpha=2,5,7} \left( \Delta_\alpha \overline{\psi}_q i\gamma_5 \tau_2 \lambda_\alpha \psi_q + \Delta_\alpha \overline{\psi}_q i\gamma_5 \tau_2 \lambda_\alpha \psi_q^c \right) - U_q ,
\end{align*}
\]
where
\[
\mu_q = \mu_q - \overline{\mu}_q = \mu_q - 2 \left[ G^2_q + G^2_q \langle \overline{\psi}_q \gamma^0 \psi_q \rangle^2 \right] \langle \overline{\psi}_q \gamma^0 \psi_q \rangle.
\]

When investigating the interplay of chiral and 2SC phase transitions, we can evaluate the phase boundaries of chiral symmetry breaking and 2SC phases at the same quark chemical potential by comparing the pressures of two phases. Here, the thermodynamic potential density is defined as \( \omega_q(T, \mu_q) = -\frac{T}{V} \ln \text{Tr} \exp \left( -\beta \int d^4 x \mathcal{H}'_q(\mu_q) \right) \) where \( V \) is a unit volume and \( \mathcal{H}'_q(\mu_q) = \mathcal{H}'_q + \mu_q \mathcal{N}_q \) with \( \mathcal{N}_q = \overline{\psi}_q \psi_q \). In Nambu-Gor’kov formalism, \( \omega_q(T, \mu_q) \) is rewritten as \( \omega_q(T, \mu_q) = -T \sum_n \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2} \ln \left( \beta S^{-1}(i\omega_n, p) \right) - U_q \) where \( S^{-1}(i\omega_n, p) \) denotes the dressed inverse Nambu-Gor’kov propagator and \( \omega_n \) are fermionic Matsubara frequencies with an integer \( n \). Thus, the
pressure \( p_q(=−\omega_q) \) is given by

\[
p_q(m_q, \mu'_q, \Delta_q; T, \mu_q) = 4 \int \frac{d^3 p}{(2\pi)^3} \left[ \sqrt{p^2 + m_q^2} + 3\sqrt{(\sqrt{p^2 + m_q^2} + \mu'_q)^2 + \Delta_q^2} \right. \\
+ 3\sqrt{(\sqrt{p^2 + m_q^2} - \mu'_q)^2 + \Delta_q^2} \sgn\left( \sqrt{p^2 + m_q^2} - \mu'_q \right) \\
+ T \ln \left( 1 + e^{-\beta(\sqrt{p^2 + m_q^2} + \mu'_q)} \right) \\
+ T \ln \left( 1 + e^{-\beta(\sqrt{p^2 + m_q^2} - \mu'_q)} \right) \\
+ 6T \ln \left( 1 + e^{-\beta(\sqrt{p^2 + m_q^2} + \mu'_q)^2 + \Delta_q^2} \right) \\
+ 6T \ln \left( 1 + e^{-\beta(\sqrt{p^2 + m_q^2} - \mu'_q)^2 + \Delta_q^2} \right) \Bigg] \\
+ G^q_q(\overline{\psi}_q\gamma^0\psi_q)^2 - G^q_q(\overline{\psi}_q\gamma^0\gamma^\mu\psi_q)^2 \\
- 3G^q_q(\overline{\psi}_q\gamma^0\psi_q)^2(\overline{\psi}_q\gamma^\mu\psi_q)^2 + \frac{3\Delta_q^2}{4G^q_q}, \tag{C.11}
\]

where

\[
\langle\overline{\psi}_q\psi_q\rangle = -4 \int \frac{d^3 p}{(2\pi)^3} \frac{m_q}{\sqrt{p^2 + m_q^2}} \left[ 1 - \left( e^{\beta(\sqrt{p^2 + m_q^2} + \mu'_q)} + 1 \right)^{-1} - \left( e^{\beta(\sqrt{p^2 + m_q^2} - \mu'_q)} + 1 \right)^{-1} \right. \\
+ \frac{3 \left( \sqrt{p^2 + m_q^2} + \mu'_q \right)}{\sqrt{\left( \sqrt{p^2 + m_q^2} + \mu'_q \right)^2 + \Delta_q^2}} \tanh \left( \frac{\beta}{2} \sqrt{\left( \sqrt{p^2 + m_q^2} + \mu'_q \right)^2 + \Delta_q^2} \right) \\
+ \left. \frac{3 \left( \sqrt{p^2 + m_q^2} - \mu'_q \right)}{\sqrt{\left( \sqrt{p^2 + m_q^2} - \mu'_q \right)^2 + \Delta_q^2}} \tanh \left( \frac{\beta}{2} \sqrt{\left( \sqrt{p^2 + m_q^2} - \mu'_q \right)^2 + \Delta_q^2} \right) \Bigg], \tag{C.12}
\]

\[
\langle\overline{\psi}_q\gamma^0\psi_q\rangle = 4 \int \frac{d^3 p}{(2\pi)^3} \left[ - \left( e^{\beta(\sqrt{p^2 + m_q^2} + \mu'_q)} + 1 \right)^{-1} + \left( e^{\beta(\sqrt{p^2 + m_q^2} - \mu'_q)} + 1 \right)^{-1} \right. \\
+ \frac{3 \left( \sqrt{p^2 + m_q^2} + \mu'_q \right)}{\sqrt{\left( \sqrt{p^2 + m_q^2} + \mu'_q \right)^2 + \Delta_q^2}} \tanh \left( \frac{\beta}{2} \sqrt{\left( \sqrt{p^2 + m_q^2} + \mu'_q \right)^2 + \Delta_q^2} \right) \\
- \left. \frac{3 \left( \sqrt{p^2 + m_q^2} - \mu'_q \right)}{\sqrt{\left( \sqrt{p^2 + m_q^2} - \mu'_q \right)^2 + \Delta_q^2}} \tanh \left( \frac{\beta}{2} \sqrt{\left( \sqrt{p^2 + m_q^2} - \mu'_q \right)^2 + \Delta_q^2} \right) \Bigg], \tag{C.13}
\]

\[
\langle\overline{\psi}_q i\gamma_5\tau_2\lambda_\alpha\psi_q\rangle = -4 \int \frac{d^3 p}{(2\pi)^3} \left[ \frac{\Delta_\alpha}{\sqrt{\left( \sqrt{p^2 + m_q^2} + \mu'_q \right)^2 + \Delta_q^2}} \tanh \left( \frac{\beta}{2} \sqrt{\left( \sqrt{p^2 + m_q^2} + \mu'_q \right)^2 + \Delta_q^2} \right) \right. \\
+ \left. \frac{\Delta_\alpha}{\sqrt{\left( \sqrt{p^2 + m_q^2} - \mu'_q \right)^2 + \Delta_q^2}} \tanh \left( \frac{\beta}{2} \sqrt{\left( \sqrt{p^2 + m_q^2} - \mu'_q \right)^2 + \Delta_q^2} \right) \Bigg], \tag{C.14}
\]
Here, $\text{sgn} \left( \sqrt{p^2 + m^2_q - \mu_q^2} \right)$ represents the sign function and $\tanh(x) = (1 - e^{-2x})(1 + e^{-2x})^{-1}$ is the hyperbolic tangent of $x$. Also, $\beta$ is the inverse temperature $T$ and $\langle \overline{\psi}_q \gamma^0 \psi_q \rangle$ corresponds to the quark number density $\rho_q$.

### C.2 Polyakov-loop potential

Approximately, we can deal with the thermodynamical aspects of quark confinement by introducing the Polyakov-loop potential $U_p(\Phi[A], \overline{\Phi}[A], T)$ in the NJL Lagrangian. This extended model is called PNJL model [22]. The two-flavor PNJL Lagrangian density with $G_{sv}$ is written as

$$
\mathcal{L}_{\text{PNJL}} = \overline{\psi}_q i\gamma^\mu D_\mu \psi_q + G^q_v \left[ \left( \overline{\psi}_q \gamma^\tau \psi_q \right)^2 + \left( \overline{\psi}_q i\gamma^5 \tau \psi_q \right)^2 \right] - G^q_v \left( \overline{\psi}_q \gamma^\mu \psi_q \right) \left( \overline{\psi}_q \gamma^\alpha \psi_q \right)
- G^q_{sv} \left[ \left( \overline{\psi}_q \gamma^\tau \psi_q \right)^2 + \left( \overline{\psi}_q i\gamma^5 \tau \psi_q \right)^2 \right] \left( \overline{\psi}_q \gamma^\mu \psi_q \right) \left( \overline{\psi}_q \gamma^\alpha \psi_q \right)
+ U_p(\Phi[A], \overline{\Phi}[A], T)
$$  \hspace{1cm} (C.15)

where the covariant derivative reads $D_\mu = \partial_\mu - i A_\mu$ with $A_\mu = \delta^a_\mu A_0$ representing background fields, in which $A_0 = g A_0 \frac{\lambda^a}{\sqrt{2}}$ being the Polyakov-gauge. Here, $g$ and $\lambda^a$ are the gauge coupling and the  \Gell-Mann\ matrices representing the color $SU(3)$-generator, respectively. The Polyakov-loop potential $U_p(\Phi, \overline{\Phi}, T)$ is an effective potential as a function of the Polyakov-loop expectation value $\Phi = \langle \text{Tr}(\ell(x)/3) \rangle$ and its conjugate $\overline{\Phi} = \langle \text{Tr}^\dagger(\ell(x)/3) \rangle$. The Polyakov loop $\ell(x)$ is a Wilson loop closed around the periodic Euclidean time direction, and defined as $\ell(x) = \mathcal{P} \exp \left[ i \int_0^\beta d\tau A_4(x, \tau) \right]$ with $A_4 = i A_0$ being the temporal gauge field and $\mathcal{P}$ the path ordering. In the Polyakov gauge, the Polyakov-loop matrix $\ell(x)$ can be written in a diagonal form in the color space. The expectation value $\Phi$ is treated as an approximate order parameter of the deconfinement phase transition. In the deconfined phase, $\Phi$ is non-zero, while it means the existence of the confined phase when $\Phi = 0$. Thus, we can clarify the interplay between the Polyakov loop and chiral dynamics.

At the mean-field level, the above Lagrangian density \hspace{1cm} (C.15) is rewritten as

$$
\mathcal{L}_{\text{PNJL}}^{\text{MF}} = \overline{\psi}_q (i\gamma^\mu \partial_\mu - m_q) \psi_q - \mu_q \overline{\psi}_q \gamma^0 \psi_q + C_q - U_p(\Phi, \overline{\Phi}, T)
$$  \hspace{1cm} (C.16)

\footnote{The sign function of $\sqrt{p^2 + m^2_q - \mu_q^2}$ is defined as follows:

$$
\text{sgn} \left( \sqrt{p^2 + m^2_q - \mu_q^2} \right) = \begin{cases} 
1 & (p^2 > \mu_q^2 - m^2_q) \\
0 & (p^2 = \mu_q^2 - m^2_q) \\
-1 & (p^2 < \mu_q^2 - m^2_q)
\end{cases}
$$

\footnote{In the pure gauge theory, the expectation value $\Phi$ is an exact order parameter of the spontaneous $Z_3$ symmetry breaking. The $Z_3$ symmetry appears as the center symmetry of the color $SU(3)$ gauge symmetry. Thus, the order parameter $\Phi$ is considered as an indicator of the deconfinement phase transition, although the $Z_3$ symmetry is not an exact symmetry in the system with dynamical quarks.}}
where

\[ m_q = -2 \left[ G^q_v - G^q_v \langle \bar{q} q \gamma^0 q \rangle \right] \langle \bar{q} q \rangle, \tag{C-17} \]
\[ \bar{\mu}_q = 2 \left[ G^q_v + G^q_v \langle \bar{q} q \gamma^0 q \rangle \right] \langle \bar{q} q \rangle, \tag{C-18} \]
\[ C_q = -G^q_v \langle \bar{q} q \gamma^0 q \rangle^2 + G^q_v \langle \bar{q} q \gamma^0 q \rangle^2 + 3G^q_v \langle \bar{q} q \gamma^0 q \rangle^2 \langle \bar{q} q \rangle^2. \tag{C-19} \]

As a result, the pressure including Polyakov loops at finite temperature and chemical potential is given as

\[
p_q(m_q, \Phi, \overline{\Phi}; T, \mu_q) = 4 \int \frac{d^3p}{(2\pi)^3} \left[ T \ln \left( 1 + 3\Phi e^{-\beta(\sqrt{p^2 + m_q^2} - \mu_q^+)} + 3\overline{\Phi} e^{-2\beta(\sqrt{p^2 + m_q^2} - \mu_q^+)} + e^{-3\beta(\sqrt{p^2 + m_q^2} - \mu_q^+)} \right) + C_q - \mathcal{U}_p(\Phi, \overline{\Phi}, T), \right. \tag{C-20} \]

Here, there are three popular ansätze for \( \mathcal{U}_p \). One is the polynomial potential [54]:

\[
\frac{\mathcal{U}_1}{T^4} = -\frac{b_2(T)}{2} \Phi \overline{\Phi} - \frac{b_3}{6} (\Phi^3 + \overline{\Phi}^3) + \frac{b_4}{4} (\Phi \overline{\Phi})^2, \tag{C-21} \]

where \( b_2(T) = a_0 + a_1(T_0/T) + a_2(T_0/T)^2 + a_3(T_0/T)^3 \) with \( a_0 = 6.75, a_1 = -1.95, a_2 = 2.625, a_3 = -7.44, b_3 = 0.75, b_4 = 7.5 \) and \( T_0 = 270 \text{ MeV} \). Another is the logarithmic ansatz [55]:

\[
\frac{\mathcal{U}_2}{T^4} = -\frac{a(T)}{2} \Phi \overline{\Phi} + b(T) \ln \left[ 1 - 6\Phi \overline{\Phi} + 4(\Phi^3 + \overline{\Phi}^3) - 3(\Phi \overline{\Phi})^2 \right], \tag{C-22} \]

with \( a(T) = a_0 + a_1(T_0/T) + a_2(T_0/T)^2 \) and \( b(T) = b_3(T_0/T)^3 \) where \( a_0 = 3.51, a_1 = -2.47, a_2 = 15.2, b_3 = -1.75 \) and \( T_0 = 270 \text{ MeV} \). The other is the ansatz inspired by a strong-coupling analysis [56]:

\[
\frac{\mathcal{U}_3}{T^4} = -bT \left[ 54e^{-a/T} \Phi \overline{\Phi} + \ln \left( 1 - 6\Phi \overline{\Phi} + 4(\Phi^3 + \overline{\Phi}^3) - 3(\Phi \overline{\Phi})^2 \right) \right], \tag{C-23} \]

where \( a = 664 \text{ MeV}, b = 0.03\Lambda_q^3 \text{ MeV}^3 \) with the momentum cutoff \( \Lambda_q \), and \( T = 200 \text{ MeV} \).

In the case with \( \mathcal{U}_p = \mathcal{U}_1 \), by minimizing \( \omega_q \) with respect to \( \langle \bar{q} q \gamma^0 q \rangle, \langle \bar{q} q \rangle, \Phi, \) and \( \overline{\Phi} \), we...
get the following relations:

\[
\langle \bar{\psi}_q \psi_q \rangle = -12 \int \frac{d^3p}{(2\pi)^3} \frac{m_q}{\sqrt{p^2 + m_q^2}} \times \left[ 1 - \frac{\Phi e^{-\beta(\sqrt{p^2 + m_q^2} + \mu_q^2)} + 2\Phi e^{-2\beta(\sqrt{p^2 + m_q^2} + \mu_q^2)} + e^{-3\beta(\sqrt{p^2 + m_q^2} + \mu_q^2)}}{1 + 3\Phi e^{-\beta(\sqrt{p^2 + m_q^2} + \mu_q^2)} + 3\Phi e^{-2\beta(\sqrt{p^2 + m_q^2} + \mu_q^2)} + e^{-3\beta(\sqrt{p^2 + m_q^2} + \mu_q^2)}} - \frac{\Phi e^{-\beta(\sqrt{p^2 + m_q^2} - \mu_q^2)}}{1 + 3\Phi e^{-\beta(\sqrt{p^2 + m_q^2} - \mu_q^2)} + 3\Phi e^{-2\beta(\sqrt{p^2 + m_q^2} - \mu_q^2)} + e^{-3\beta(\sqrt{p^2 + m_q^2} - \mu_q^2)}} \right],
\]

(C-24)

\[
\langle \bar{\psi}_q \gamma^0 \psi_q \rangle = \rho_q
\]

\[
= 12 \int \frac{d^3p}{(2\pi)^3} \left[ -\frac{\Phi e^{-\beta(\sqrt{p^2 + m_q^2} + \mu_q^2)} + 2\Phi e^{-2\beta(\sqrt{p^2 + m_q^2} + \mu_q^2)} + e^{-3\beta(\sqrt{p^2 + m_q^2} + \mu_q^2)}}{1 + 3\Phi e^{-\beta(\sqrt{p^2 + m_q^2} + \mu_q^2)} + 3\Phi e^{-2\beta(\sqrt{p^2 + m_q^2} + \mu_q^2)} + e^{-3\beta(\sqrt{p^2 + m_q^2} + \mu_q^2)}} + \frac{\Phi e^{-\beta(\sqrt{p^2 + m_q^2} - \mu_q^2)}}{1 + 3\Phi e^{-\beta(\sqrt{p^2 + m_q^2} - \mu_q^2)} + 3\Phi e^{-2\beta(\sqrt{p^2 + m_q^2} - \mu_q^2)} + e^{-3\beta(\sqrt{p^2 + m_q^2} - \mu_q^2)}} \right],
\]

(C-25)

\[
T^3 \left( -b_2(T) \Phi - b_3 \Phi^2 + b_4 \Phi^3 \right)
\]

\[
= 24 \int \frac{d^3p}{(2\pi)^3} \left[ \frac{e^{-\beta(\sqrt{p^2 + m_q^2} + \mu_q^2)}}{1 + 3\Phi e^{-\beta(\sqrt{p^2 + m_q^2} + \mu_q^2)} + 3\Phi e^{-2\beta(\sqrt{p^2 + m_q^2} + \mu_q^2)} + e^{-3\beta(\sqrt{p^2 + m_q^2} + \mu_q^2)}} e^{-2\beta(\sqrt{p^2 + m_q^2} - \mu_q^2)} \right],
\]

(C-26)

\[
T^3 \left( -b_2(T) \Phi - b_3 \Phi^2 + b_4 \Phi^3 \right)
\]

\[
= 24 \int \frac{d^3p}{(2\pi)^3} \left[ \frac{e^{-2\beta(\sqrt{p^2 + m_q^2} + \mu_q^2)}}{1 + 3\Phi e^{-\beta(\sqrt{p^2 + m_q^2} + \mu_q^2)} + 3\Phi e^{-2\beta(\sqrt{p^2 + m_q^2} + \mu_q^2)} + e^{-3\beta(\sqrt{p^2 + m_q^2} + \mu_q^2)}} \right],
\]

(C-27)

### C.3 Tensor-type interaction

In nature, very high-density systems may correspond to the deep interior of compact stars such as neutron stars. Thus, from observations of these stars, it is expected that the constraints on cold high-density matter are obtained. In particular, the neutron stars with very large magnetic fields on the order of $10^{15}$ Gauss at their surface, the so-called magnetars, are of particular interests since there has recently been growing evidence for the existence of the magnetars \[67\]. Until now, concentrating on the high-density quark matter, a possibility of quark ferromagnetism or quark spin polarization has been considered \[68\]. In this context, a coexistent phase of spin polarization and color superconductivity is investigated by introducing the one-gluon exchange interaction as an effective interaction between quarks \[69\], and a relation between the quark spin polarization
and the spontaneous chiral symmetry breaking is also studied by using an NJL-type model with axial-vector interaction \[^4\], in which the spin-polarized quark phase occurs when the expectation value for the pseudovector bilinear becomes non-zero, i.e., \( \langle \overline{\psi}_q^t \gamma_5 \gamma_\nu \psi_q \rangle = \langle \overline{\psi}_q^t \Sigma \gamma_\nu \psi_q \rangle \neq 0 \). However, the mechanism for generating such strong magnetic fields has not yet been established.

Recently, the possibility of spin polarization in high-density symmetric quark matter is indicated\[^5\] using an NJL-type model including tensor-type interaction between quarks \[^6\]\[^6\], and the stability of the spin-polarized quark phase against the 2SC phase is also investigated \[^6\]. Here, the tensor term in Lagrangian density is as follows:

\[
\mathcal{L}_t = -G_t^2 \left[ \langle \overline{\psi}_q^t \gamma_\mu \gamma_\nu \tau_k \psi_q \rangle (\overline{\psi}_q \gamma_\mu \gamma_\nu \tau_k \psi_q) + (\overline{\psi}_q \gamma_5 \gamma_\mu \gamma_\nu \psi_q \rangle (\overline{\psi}_q \gamma_5 \gamma_\mu \gamma_\nu \psi_q) \right],
\]

where \( G_t^2 \) represents the coupling constant of the tensor-type four-point interaction\[^6\]. When investigating the spontaneous spin polarization of symmetric quark matter, the realized phase is determined by comparing the pressures of the quark spin polarized phase and the normal phase at finite temperature and quark chemical potential \[^6\]. The pressure, which is derived from the effective potential \( V_{\text{eff}} \), is given by

\[
p_q(m_q = 0, F_k; T, \mu_q) = -V_{\text{eff}}(\langle \overline{\psi}_q \gamma_\nu \psi_q \rangle = 0, F_k \equiv -4G_t^2 \langle \overline{\psi}_q \Sigma \gamma_\mu \tau_k \psi_q \rangle; T, \mu_q) \]

\[
= -6 \int F_k \int d^3 p \int \frac{d^3 p}{(2\pi)^3} \left[ \frac{F_k - \sqrt{p_1^2 + p_2^2}}{\sqrt{p_1^2 + p_2^2}} \left( n_+^{(-)} - n_-^{(-)} \right) \right.
\]

\[
+ \frac{F_k + \sqrt{p_1^2 + p_2^2}}{\sqrt{p_1^2 + p_2^2}} \left( n_+^{(+)} - n_-^{(+)} \right) \left. \right] - \frac{F_k^2}{8G_t^2},
\]

where \( n_\pm^{(\pm)} \) being Fermi distribution function\[^6\] is defined by

\[
n_\pm^{(\pm)} = \left[ e^{\beta \left( \pm \sqrt{\langle F_k \tau_k \rangle \pm \sqrt{p_1^2 + p_2^2}^2 + p_3 - \mu_q \rangle} \right) + 1 \right]^{-1},
\]

in which the number of color is taken as three, and the chiral condensate \( \langle \overline{\psi}_q \gamma_\nu \psi_q \rangle \) and the excited modes are discarded since the vacuum at high-density quark matter is only considered. Here, \( F_k \) corresponds to the quark spin polarization or the quark fermromagnetic condensate in the tensor mean field. Also, we keep only \( F_{k=3} \) since the expectation values \( \langle \tau_{k=1,2} \rangle \) for u and d quark fields

\[^4\] Although it was pointed out that the quark spin polarization was realized using the axial-vector interaction \[^5\], it was shown that the quark spin alignment disappears when the quark mass is zero in the chiral symmetric phase \[^5\]. However, even in the chiral symmetric phase, it was indicated that the quark spin polarization occurs.

\[^5\] In the Dirac representation for the Dirac gamma matrices, \( \gamma_\mu \gamma_\nu = -i \Sigma_3 = -i \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \), where \( \sigma_3 \) denotes the third component of the Pauli matrix. If the vacuum expectation value of \( \overline{\psi}_q \gamma_5 \gamma_\nu \tau_k \psi_q \) is finite, the spin of quarks is aligned along the third axis, which leads to the quark spin polarization or the quark magnetization. Thus, it is sufficient to investigate a possibility of spin alignment along the third axis.

\[^6\] It should be replace \( n_\pm^{(\pm)} \) into \( n_\pm^{(\pm)} - 1 \) if the negative energy contribution is omitted. When taking account to the negative energy contribution, it is possible to introduce the three momentum cutoff \( \Lambda_q \).
are zero, which leads to $F_{k=1,2} = 0$. The expectation value $F_k = -4G^2 \langle \bar{\psi}_q \Sigma_3 \tau_k \psi_q \rangle$ for the quark spin polarization is derived by minimizing the effective potential $V_{\text{eff}}$ with respect to $F_k$. Here, the expectation value $\langle \bar{\psi}_q \Sigma_3 \tau_k \psi_q \rangle$ is given by

\[
\langle \bar{\psi}_q \Sigma_3 \tau_k \psi_q \rangle = 6 \int \frac{d^3 p}{(2\pi)^3} \left[ \frac{F_k - \sqrt{p_1^2 + p_2^2}}{\sqrt{p^2 + F_k^2 - 2F_k \sqrt{p_1^2 + p_2^2}}} \left( n_+^{(-)} - n_-^{(-)} \right) + \frac{F_k + \sqrt{p_1^2 + p_2^2}}{\sqrt{p^2 + F_k^2 + 2F_k \sqrt{p_1^2 + p_2^2}}} \left( n_+^{(+)} - n_-^{(+)} \right) \right].
\]  

(C.31)
APPENDIX D

Approach to $\beta$-stable stellar matter

D.1 The equilibrium configuration of compact stars

Compact stars such as neutron stars are bound by the gravity, not the nuclear strong force, since the stars are stabilized by the internal pressure, i.e., the degeneracy pressure of neutrons, against the gravitational collapse\(^1\). Indeed, the compact star composed of only neutral and positive charged particles are unstable, so that the charge neutral condition and leptons, e.g., electrons, are required. In the case where the electron’s Fermi energy reaches the muon rest mass, heavy electrons, i.e., muons are considered. The muon $\mu^-$ decays to the electron $e^-$, the electron anti-neutrino $\bar{\nu}_e$, and the muon neutrino $\nu_\mu$, i.e., $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$. The corresponding chemical equilibrium condition is given by $\mu_e = \mu_\mu$ where $\mu_e$ represents the electron chemical potential and $\mu_\mu$ denotes the muon chemical potential. For the charge neutrality, the proton number density $\rho_p$ is equivalent to the lepton number density $\rho_l$, i.e., $\rho_p = \rho_l = \rho_e + \rho_\mu$ where $\rho_e$ and $\rho_\mu$ are the number density of the electron and the muon, respectively. The baryon number conservation reads $\rho_B = \rho_n + \rho_p$ where $\rho_B$, $\rho_n$, and $\rho_p$ denote the baryon number density, the neutron number density, and the proton number density, respectively. Here, one of the effect of neutrinos are related to the thermal evolution, i.e., the cooling of compact stars\(^2\).

Toward the interior of the star, i.e., as the density increases, the matter becomes neutron-rich which is the energetically favored state via the inverse $\beta$-decay ($p + e^- \to n + \nu_e$). In the crust of the

---

1 If the electron chemical potential exceeds the muon rest mass ($\mu_e > m_\mu = 105.7$ MeV), the $npe\mu$ matter, which is the neutron star matter with the equilibrium configuration of neutrons, protons, electrons, and muons, is realized, as the neutron density increases. On the other hand, as the baryon density increases, the new particles called hyperons appear in the case where the neutron’s Fermi energy exceeds the neutron decay threshold into the hyperon which is the baryon containing strange quarks, e.g., $\Lambda$ and $\Sigma$.

2 The process of the neutrino emission depends strongly on the matter in the interior of the star. In the nuclear matter case, the neutrino emission process is allowed only in the modified Urca process ($n + n \to n + p + e^- + \bar{\nu}_e$, $n + p + e^- \to n + n + \nu_e$) since the direct Urca process ($n \to p + e^- + \bar{\nu}_e$, $p + e^- \to n + \nu_e$) is forbidden due to the neutron excess. This process, however, has the low neutrino emissivity. Therefore, the more powerful neutrino emission is required nowadays. The possible scenario is the neutrino emission in the quark matter. In this speculative scenario, there may be too powerful to produce the neutrino energy loss, but it is possible to suppress the neutrino emission by taking into account the color superconducting states.
star, neutrons are unstable since the neutron-drip line where the neutron leaches from the nuclei is reached. In the inner crust permeated by free neutron fluids, nuclei are considered to be closely packed in a Coulomb lattice. In this region, the phases with inhomogeneous structure, the so-called pasta phases regarded as liquid-gas mixed phases, may appear due to the competition between the Coulomb interaction and the surface tension. For one of the most recent review, see e.g., Ref. [26].

In the neutron star interior, the equilibrium state would be $\beta$-stable, i.e., $n \leftrightarrow p + e^-$ where the neutrinos are ignored, since the lifetime of free neutrons is shorter than that of the star, i.e., the neutron does not undergo the $\beta$-decay ($n \rightarrow p + e^- + \nu_e$). This leads to the condition $\mu_n = \mu_p + \mu_e$ where $\mu_n$ and $\mu_p$ represent the neutron chemical potential and the proton chemical potential, respectively. Therefore, the equilibrium configuration of $n$, $p$, and $e^-$ is determined by the chemical equilibrium ($\mu_n = \mu_p + \mu_e$), the charge neutrality ($\mu_p = \mu_e$), and the conservation of the baryon number density ($\rho_B = \rho_n + \rho_p$).

D.2 Application to quark stellar matter

We consider the degenerate free quarks with $u$ and $d$ flavors and electrons $e^-$. In the $\beta$-stable quark matter with electric and color neutralities, the quark beta decay process and its inverse one occur at identical rates. Under the $\beta$ equilibrium ($d \leftrightarrow u + e^-$), the charge neutrality, and the baryon number conservation, the following conditions are imposed:

$$\mu_d = \mu_u + \mu_e, \quad (D.1)$$

$$\rho_e = (2\rho_u - \rho_d)/3, \quad (D.2)$$

$$\rho_B = (\rho_u + \rho_d)/3, \quad (D.3)$$

where $\mu_u$, $\mu_d$, $\rho_u$, and $\rho_d$ represent the $u$-quark chemical potential, the $d$-quark chemical potential, the $u$-quark number density, and the $d$-quark number density, respectively.

We are concerned here with quark stars composed of pure two-flavor quark matter. Here, we treat the final stages of the star evolution with maximum entropy. Thus, we can set the neutrino chemical potential to zero since neutrinos have already diffused out. For the quark stellar matter, the equation of state (EoS) in the extended NJL model with scalar-vector eight-point interactions (see [21]) may be available, introducing the lepton contribution to the energy density and pressure. Here, the color and electric neutrality constraints with $\beta$-equilibrium are given by Eqs. (D.2) and (D.3) and also the lepton EoS is required. With the aid of the quark EoS, $p = p_q(\epsilon)$, we can solve

---

3 The feasibility of pasta structures in quark stars has recently been investigated, taking account of the effects of surface tension and electric charge Debye screening [25].

4 In the case where the quark matter occurs inside the compact star, it is necessary to support the high mass constraint. Thus, the quark star or the hybrid star with quark core requires the sufficient stiff EoS. Here, the repulsive vector interaction leads to a stiffer quark EoS. For the too stiff EoS, it is possible to fine-tune with the scalar-vector coupling.
the Tolman-Oppenheimer-Volkoff (TOV) equation \([67, 68]\) based on general relativity:

\[
\frac{dp}{dr} = - \frac{G}{r^2} \frac{(\mathcal{M}(r) + 4\pi r^3 p) (\epsilon + p)}{r (r - 2GM(r))} ,
\]

\[
\mathcal{M}(r) = 4\pi \int_0^r \epsilon r^2 dr ,
\]

where \(G\), \(\mathcal{M}(r)\), and \(\epsilon\) are the gravitational constant or Newton’s constant, the mass inside radius \(r\), and the energy density. These are the first-order coupled differential equations integrated from the initial state where \(\mathcal{M}(r = 0)\) and \(\epsilon(r = 0) = \epsilon_c\) with \(\epsilon_c\) being the central energy density to surface where \(p(R) = 0\) with \(R\) being the radius of stars. Thus, the TOV equation describes the global structure of a static and spherically symmetric non-rotating stars. Consequently, the relation between the star mass and its central energy density is obtained, which leads to the mass-radius (MR) relation for quark stars.

Indeed, as a more realistic system, the quark star with three flavors, i.e., strange star\([69]\), is considered\([69]\). The equilibration conditions for the \(\beta\)-stable \(udse\) matter, in which \(d \leftrightarrow u + e^-, s \leftrightarrow u + e^-, \) and \(d + u \leftrightarrow u + s\), are given by

\[
\mu_d = \mu_u + \mu_e ,
\]

\[
\rho_s = \mu_u ,
\]

\[
0 = 2\rho_n/3 - \rho_d/3 - \rho_s/3 - \rho_e ,
\]

\[
\rho_B = (\rho_u + \rho_d + \rho_s)/3 ,
\]

where \(\rho_s\) denotes the strange quark number density. The extremely high-density regime, i.e., \(\mu_q \gg m_q\), or the massless case leads Eqs. (D.7) and (D.8) to \(\mu_u = \mu_d = \mu_s\) and \(\mu_e = 0\) where \(\mu_q\) represents the strange quark chemical potential. This implies that the massless quark matter has charge neutrality without electrons while the massive case with \(m_s\) needs the electrons so as to charge neutral the system. In the Lagrangian of three-flavor system, the Kobayashi-Maskawa’t Hooft (KMT) six-quark interaction with \(U(1)_A\) anomaly is required:

\[
\mathcal{L}_{KTM} = -G_k^7 \left[ \det_{ij} \bar{\psi}_i (1 + \gamma_5) \psi_j + h.c. \right] ,
\]

where \(\psi_i (i = u, d, s)\) is the quark field with three flavors and three colors. Also, the superconducting quark phase should be considered for the description of hybrid stars or quark stars. Moreover, as the phenomenological implications, e.g., for compact stars, the nucleon superfluidity (3\(P_2\) Cooper pairs), the meson condensation (pion and kaon condensations), the spatially nonuniform chiral condensate (e.g., the dual chiral density wave\([70]\) have been proposed recently. In particular, the information of the quark matter EoS with quark pasta or chiral inhomogeneous structures may contribute to the magnetic and cooling effects on quark stars or hybrid stars.

\footnote{There is the strange matter hypothesis in which the quark matter contains roughly equal numbers of \(u, d, \) and \(s\) quarks. The large pieces of the strange matter is called strange stars, while the small nuggets of the strange matter is called strangelets suggested as a dark matter candidate.}

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References


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